Slope Fields

Long ago, in a class room not so very far away, you learned about a mathematical idea called the **derivative**. Now assuming that you passed the course you were taking back then, you should have learned a very important property of the derivative:

the *derivative* of a function gives its *slope*

When we work with **differential equations**, we are dealing with expressions in which the derivative appears as a variable. For example, we might be asked to analyze the differential equation:

$$dy/dx = x^2$$

If we simply replace the variable *dy/dx* in the above equation by what we learned it means from our calculus course, we get the following statement:

$$slope = x^2$$

So what? Well, let's remind ourselves of our usual **goal** when we are given a differential equation:

find the function whose derivative appears in the equation

In our example this means that our goal is to find y.

Now you may be one of those clever students who's always one step ahead of the instructor. If so you're probably already having thoughts about how you could easily solve the current example (i.e. find y) using integration. Hold that thought! Unfortunately integration isn't something we will always be able to use. Many (most?) differential equations *can't* be integrated directly. What we're leading into here is a method that can help us on far more differential equations than can be solved using integration.

It is possible to learn a great deal about a differential equation, even when we don't know how to solve the equation, by looking at a picture of its *trajectory* space. In particular, a solution is unique if no two curves in trajectory space intersect. It is actually quite easy to visualize a trajectory space by sketching the ***slope field*** (sometimes called a vector field, flow field, or direction field) of the differential equation. *The slope field* of the differential equation

$$y = \frac{dy}{dx}$$

is the vector field defined in the following manner: to every point (x, y) in the domain of f, assign a unit vector with slope $f'(x,y) = \left.\frac{dy}{dx}\right|_{(x,y)}$. The vector will then point in a direction tangent to a trajectory that passes through the point (x, y) .

Anyway, let's get back to our analysis of slope. We've established that our goal is to find the function which satisfies:

$$slope = x^2$$

In other words, we're seeking a function whose slope at any point in the (x,y)-plane is equal to the value of x^2 at that point. Let's amplify that by examining a few selected points.

- At the point (1,2) the slope would be $1^2 = 1$.
- At the point (5,3) the slope would be $5^2 = 25$.
- At the point (-3,11) the slope would be $(-3)^2 = 9$.

(Notice that the y-value of these points doesn't influence the slope in this particular example. This will not always be the case.)

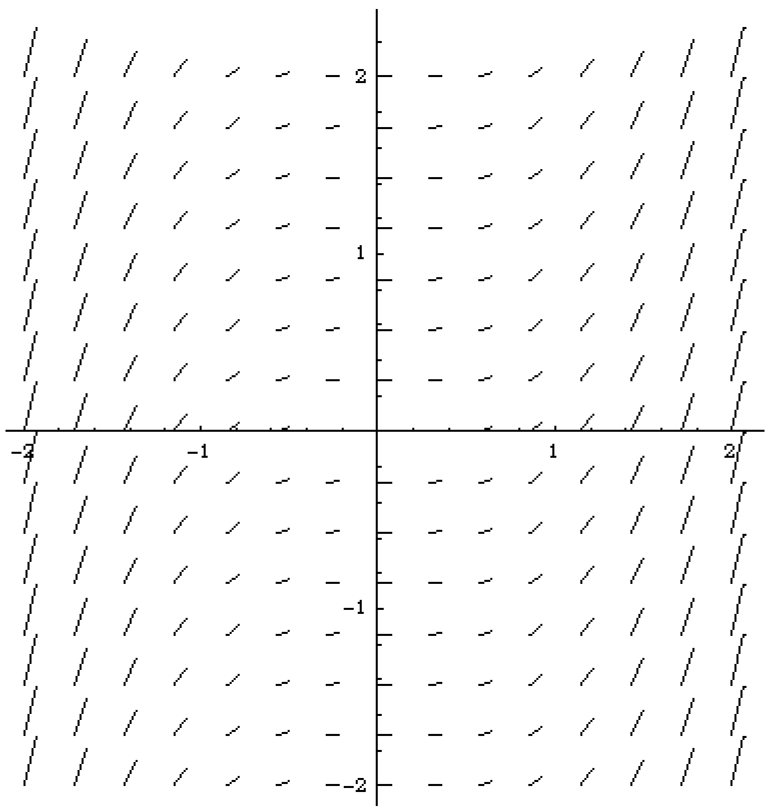
Hmmm...maybe we could use these slopes to get a picture of what the function we seek--the function which has these slopes--looks like? Well, for starters we'd need to be a bit more systematic about how we choose our points. (The choices I made above were somewhat random.)

We could divide the entire plane into a grid, kind of like the squares on a piece of graph paper, and at each grid intersection we could make a slope calculation like we were doing above. Obviously doing this for the entire plane is actually impossible, since it's infinite, so we'll have to be satisfied with some "reasonable" subset of the plane.

This is starting to sound like a lot of work. We may be talking about slope calculations at literally thousands of points, here. Sounds like a job for someone who doesn't mind doing myriads of mind-numbingly repetitive tasks. Someone who can maintain accuracy despite the mountain of (admittedly trivial) calculations involved. Well, you knew you were sitting at a computer for a reason, didn't you?

OK, so we'll have the computer do the calculations, but there's still something we haven't decided on yet! What do we do with all those thousands of slopes once we've found them? We mentioned earlier that we'd use the slopes to get a picture of what the function *y* looks like. One way of doing this would be to graphically represent each of the slopes that we find at points all over the plane by a short line segment that is actually as steep as the slope says it should be at that point. We can think of these little line segments as tangent lines to the function *y* that we've been looking for all this time. We will, of course, have the computer also carry out the job of drawing all the little tangent lines for us. We call the resulting picture a ***slope field***.

The picture produced by a particular computer program may look a little like this:



A sample slope field made with Mathematica

Let's try to make one:

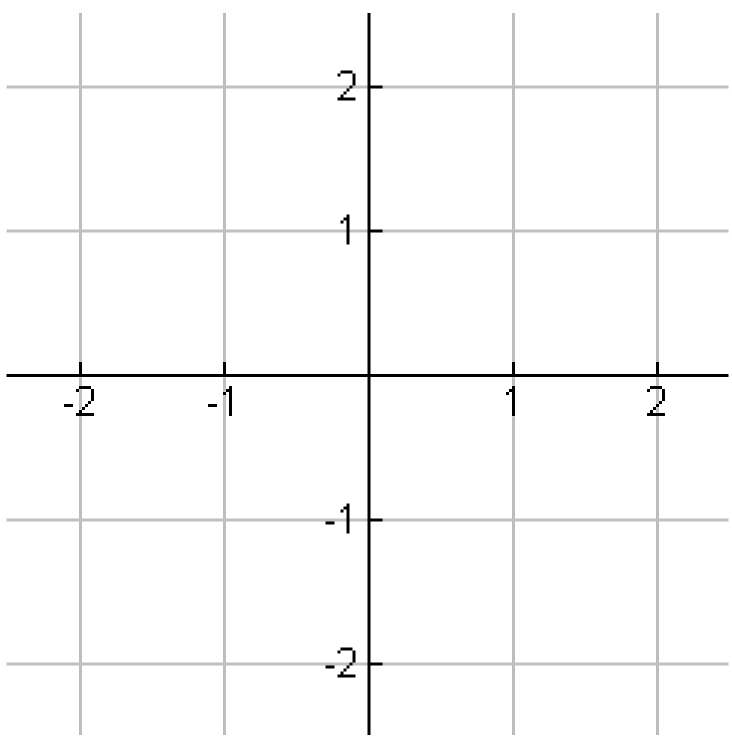
Ex:

Find a slope field for the differential equation:

$$dy/dx = x^2$$

As I mentioned above, it would be impossible to produce a slope field covering the entire, infinite, Cartesian plane. Instead, for our example, let's restrict the section of the plane we consider to the following intervals: $-2 < x < 2$, and $-2 < y < 2$ using the integer coordinates.

$\left.\frac{dy}{dx}\right\|_{(-2,2)} =$	$\left.\frac{dy}{dx}\right\|_{(-1,2)} =$	$\left.\frac{dy}{dx}\right\|_{(0,2)} =$	$\left.\frac{dy}{dx}\right\|_{(1,2)} =$	$\left.\frac{dy}{dx}\right\|_{(2,2)} =$
$\left.\frac{dy}{dx}\right\|_{(-2,1)} =$	$\left.\frac{dy}{dx}\right\|_{(-1,1)} =$	$\left.\frac{dy}{dx}\right\|_{(0,1)} =$	$\left.\frac{dy}{dx}\right\|_{(1,1)} =$	$\left.\frac{dy}{dx}\right\|_{(2,1)} =$
$\left.\frac{dy}{dx}\right\|_{(-2,0)} =$	$\left.\frac{dy}{dx}\right\|_{(-1,0)} =$	$\left.\frac{dy}{dx}\right\|_{(0,0)} =$	$\left.\frac{dy}{dx}\right\|_{(1,0)} =$	$\left.\frac{dy}{dx}\right\|_{(2,0)} =$
$\left.\frac{dy}{dx}\right\|_{(-2,-1)} =$	$\left.\frac{dy}{dx}\right\|_{(-1,-1)} =$	$\left.\frac{dy}{dx}\right\|_{(0,-1)} =$	$\left.\frac{dy}{dx}\right\|_{(1,-1)} =$	$\left.\frac{dy}{dx}\right\|_{(2,-1)} =$
$\left.\frac{dy}{dx}\right\|_{(-2,-2)} =$	$\left.\frac{dy}{dx}\right\|_{(-1,-2)} =$	$\left.\frac{dy}{dx}\right\|_{(0,-2)} =$	$\left.\frac{dy}{dx}\right\|_{(1,-2)} =$	$\left.\frac{dy}{dx}\right\|_{(2,-2)} =$



<u>A summary of the steps involvedto draw a slope field</u>:

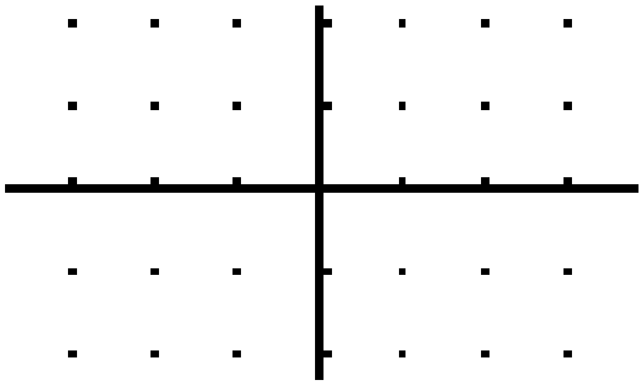
This procedure may be used to make pictures of slope fields by hand, however, a far less tedious option is to have the computer follow these steps for you.

1. Put the differential equation in the form $dy/dx = f'(x,y)$.
2. Decide upon what rectangular region of the plane you want to make the picture.
3. Impose a grid on this region.
4. Calculate the value of the slope, $f'(x,y)$, at each grid point, (x,y).
5. Sketch a picture in which at each grid point there is a short line segment having the corresponding slope.

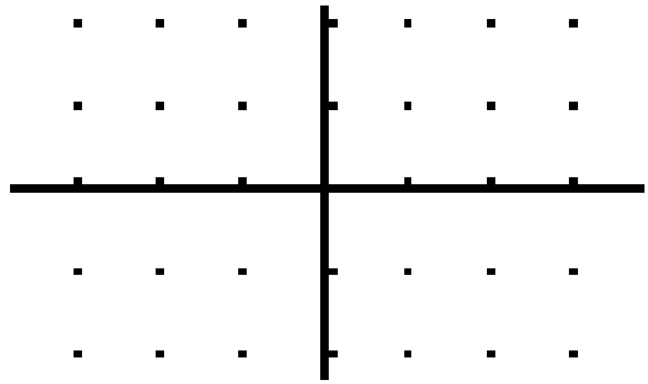
WORKSHEET ON SLOPE FIELDS

Draw a slope field for each of the following differential equations.

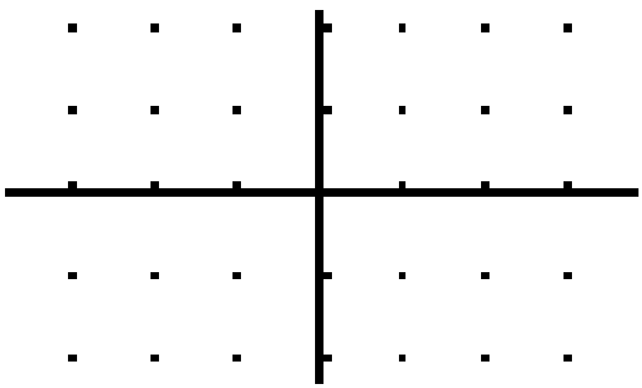
1. $\frac{dy}{dx} = x + 1$



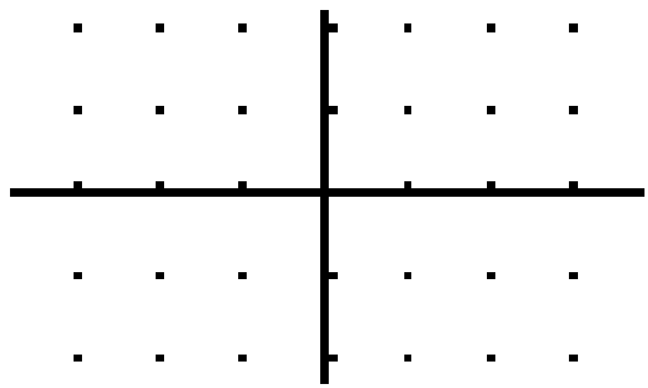
2. $\frac{dy}{dx} = 2y$



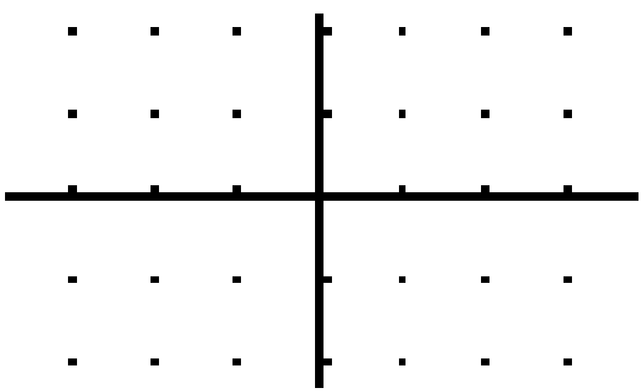
3. $\frac{dy}{dx} = x + y$



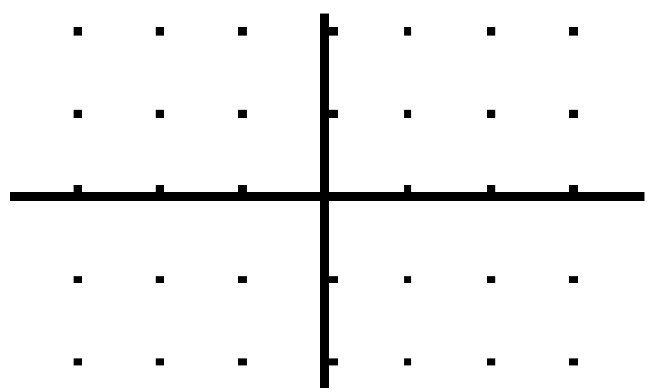
4. $\frac{dy}{dx} = 2x$



5. $\frac{dy}{dx} = y - 1$

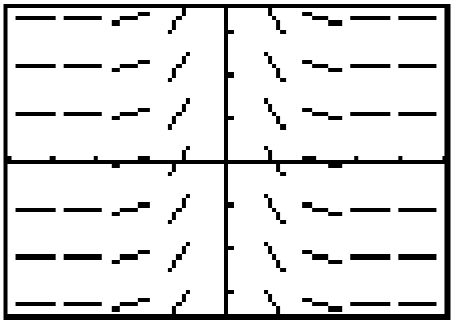


6. $\frac{dy}{dx} = -\frac{y}{x}$

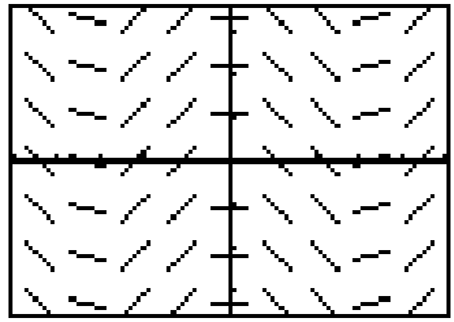


For 7 – 14, match each slope field with the **equation** that the slope field could represent.

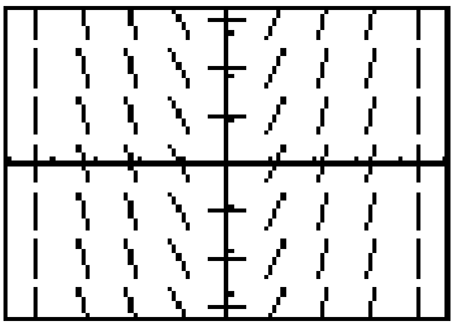
(A)



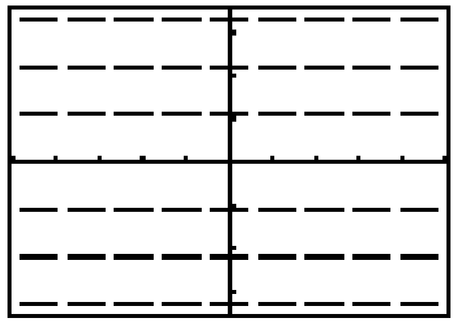
(B)



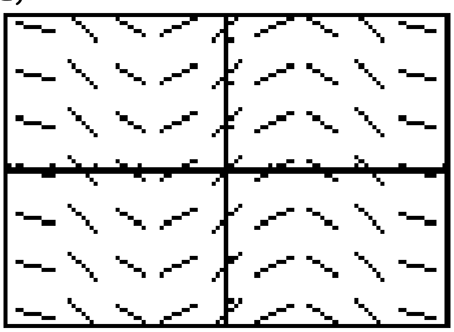
(C)



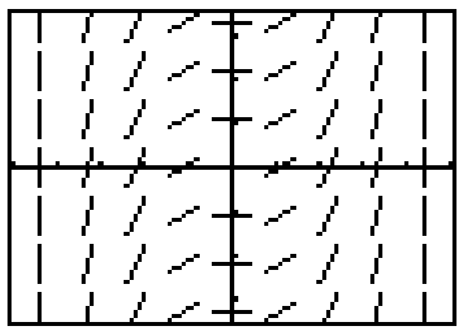
(D)



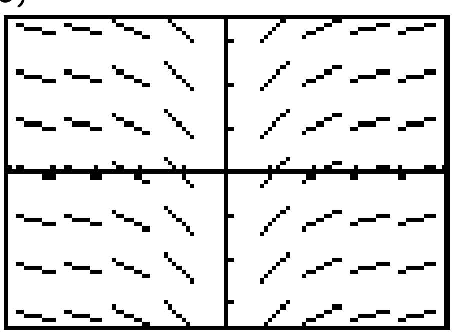
(E)



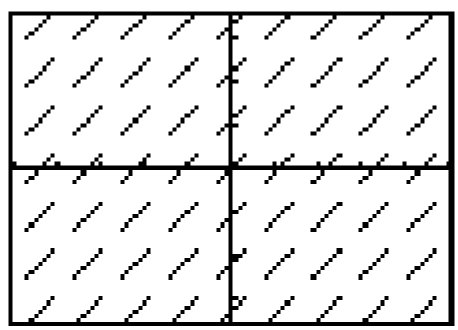
(F)



(G)



(H)



7. $y = 1$

8. $y = x$

9. $y = x^2$

10. $y = \frac{1}{6}x^3$

11. $y = \frac{1}{x^2}$

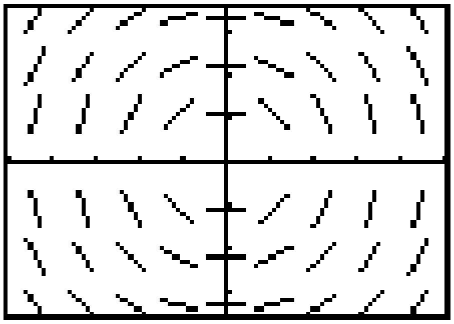
12. $y = \sin x$

13. $y = \cos x$

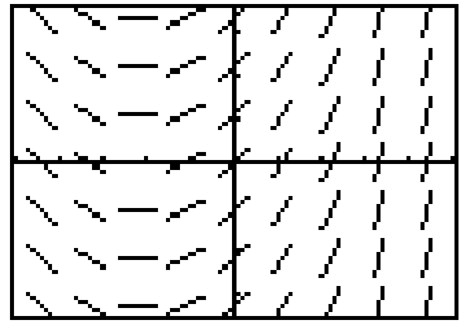
14. $y = \ln|x|$

Match the slope fields with their differential equations.

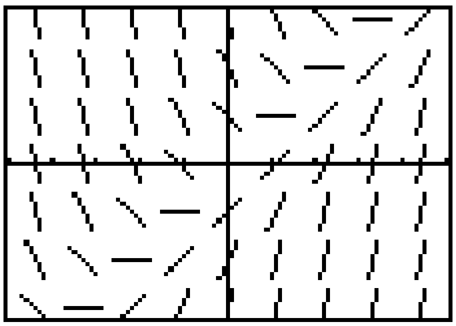
(A)



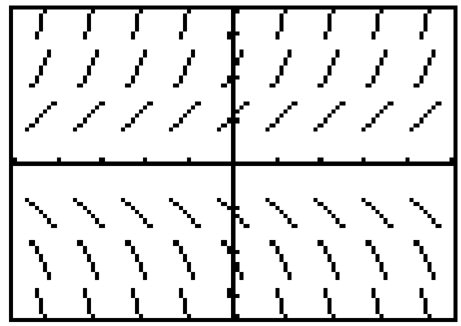
(B)



(C)



(D)



15. $\frac{dy}{dx} = \frac{1}{2}x + 1$

16. $\frac{dy}{dx} = x - y$

17. $\frac{dy}{dx} = y$

18. $\frac{dy}{dx} = -\frac{x}{y}$

19. The calculator-drawn slope field for the differential equation $\frac{dy}{dx} = xy$ is shown in the figure below. The solution curve passing through the point $(0,1)$ is also shown.

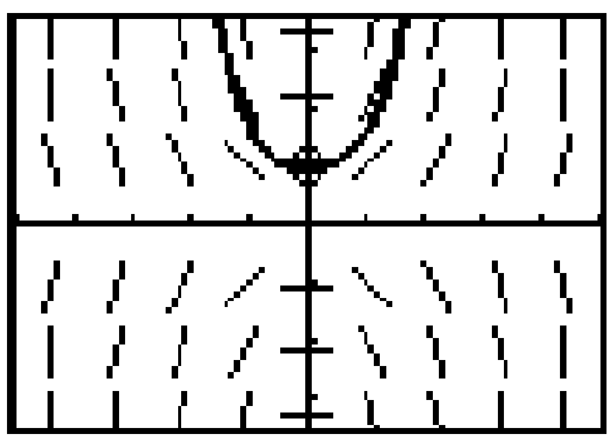
(A) Find the general solution for $\frac{dy}{dx}$.

(B) Find the particular solution for the one shown, passing through $(0,1)$.

(C) Sketch this solution on your calculator on the windows X[-5,5], Y[-5,5]. Does it look like the drawn solution?

(D) Sketch the solution curve through the point $(0,2)$.

(E) Sketch the solution curve through the point $(0,-1)$.

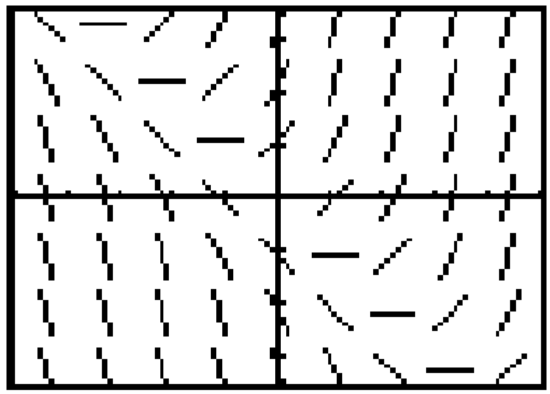


20. The calculator-drawn slope field for the differential equation $\frac{dy}{dx} = x + y$ is shown in the figure below.

(A) Sketch the solution curve through the point $(0,1)$.

(B) Sketch the solution curve through the point $(-3,0)$.

(C) Approximate $y(-3.1)$ using the equation of the tangent line to $y = f(x)$ at the point $(-3,0)$.

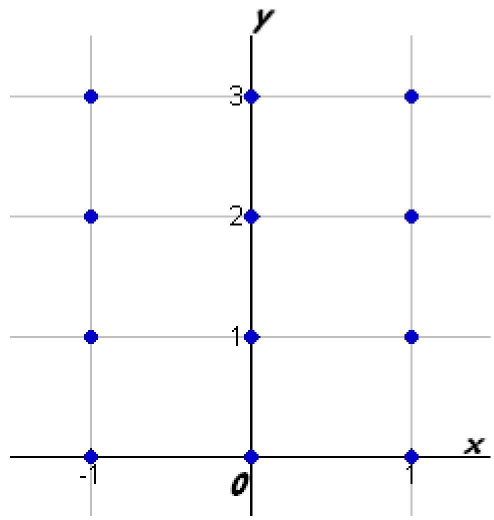


Sample AP Free-Response Questions

2004 AB-6 (No Calculator)

Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

(A) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.



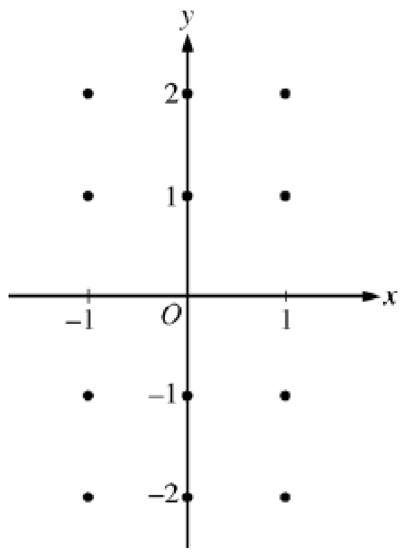
(B) While the slope field in part (A) is drawn at only twelve points, it is defined at every point in the *xy*-plane. Describe all points in the *xy*-plane for which the slopes are positive.

(C) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$.

2005 AP Calculus AB-6 Free-Response Question
(NO CALCULATOR)

Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated. **(Note: Use the axes provided in the pink test booklet.)**



(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.