SECTION I

Part A  TIME: 55 MINUTES

The use of calculators is not permitted for this part of the examination. There are 28 questions in Part A, for which 55 minutes are allowed. Because there is no deduction for wrong answers, you should answer every question, even if you need to guess.

Directions: Choose the best answer for each question.

1. \( \lim_{x \to \infty} \frac{20x^2 - 13x + 5}{5 - 4x^3} \) is
   (A) \(-5\)  (B) \(\infty\)  (C) 0  (D) 5  (E) 1

2. \( \lim_{h \to 0} \frac{\ln(2+h) - \ln 2}{h} \) is
   (A) 0  (B) \(\ln 2\)  (C) \(\frac{1}{2}\)  (D) \(\frac{1}{\ln 2}\)  (E) \(\infty\)

3. If \( x = \sqrt{1-t^2} \) and \( y = \sin^{-1} t \), then \( \frac{dy}{dx} \) equals
   (A) \(-\frac{\sqrt{1-t^2}}{t}\)  (B) \(-t\)  (C) \(\frac{t}{1-t^2}\)  (D) 2  (E) \(\frac{1}{t}\)

Questions 4 and 5. Use the following table, which shows the values of the differentiable functions \( f \) and \( g \).

<table>
<thead>
<tr>
<th>x</th>
<th>( f )</th>
<th>( f' )</th>
<th>( g )</th>
<th>( g' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>(\frac{1}{2})</td>
<td>-3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>4</td>
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<tr>
<td>4</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>(\frac{1}{2})</td>
</tr>
</tbody>
</table>
4. The average rate of change of function $f$ on $[1,4]$ is
   (A) $\frac{7}{6}$  (B) $\frac{4}{3}$  (C) $\frac{15}{8}$  (D) $\frac{9}{4}$  (E) $\frac{8}{3}$

5. If $h(x) = g(f(x))$ then $h'(3) =$
   (A) $\frac{1}{2}$  (B) $1$  (C) $4$  (D) $6$  (E) $9$

6. $\int_1^2 (3x - 2)^3\ dx$ is equal to
   (A) $\frac{16}{3}$  (B) $\frac{63}{4}$  (C) $\frac{13}{3}$  (D) $\frac{85}{4}$  (E) none of these

7. If $y = \frac{x - 3}{2 - 5x}$, then $\frac{dy}{dx}$ equals
   (A) $\frac{\frac{17 - 10x}{(2 - 5x)^2}}{}$  (B) $\frac{\frac{13}{(2 - 5x)^2}}{}$  (C) $\frac{\frac{x - 3}{(2 - 5x)^2}}{}$
   (D) $\frac{\frac{17}{(2 - 5x)^2}}{}$  (E) $\frac{\frac{-13}{(2 - 5x)^2}}{}$

8. The maximum value of the function $f(x) = xe^{-x}$ is
   (A) $\frac{1}{e}$  (B) $e$  (C) $1$  (D) $-1$  (E) none of these

9. Which equation has the slope field shown below?

   (A) $\frac{dy}{dx} = \frac{5}{y}$  (B) $\frac{dy}{dx} = \frac{5}{x}$  (C) $\frac{dy}{dx} = \frac{x}{y}$
   (D) $\frac{dy}{dx} = 5y$  (E) $\frac{dy}{dx} = x + y$
Questions 10–11. The graph below shows the velocity of an object moving along a line, for $0 \leq t \leq 9$.

10. At what time does the object attain its maximum acceleration?
   (A) $2 < t < 5$  (B) $5 < t < 8$  (C) $t = 6$  (D) $t = 8$  (E) $8 < t < 9$

11. The object is farthest from the starting point at $t =$
   (A) 2  (B) 5  (C) 6  (D) 8  (E) 9

12. If $x = 2 \sin \theta$, then $\int_{0}^{\pi/2} \frac{x^2 dx}{\sqrt{4 - x^2}}$ is equivalent to:
   (A) $4 \int_{0}^{\pi/2} \sin^2 \theta d\theta$  (B) $\int_{0}^{\pi/2} 4 \sin^2 \theta d\theta$  (C) $\int_{0}^{\pi/2} 2 \sin \theta \tan \theta d\theta$
   (D) $\int_{0}^{\pi/2} \frac{2 \sin^2 \theta}{\cos^2 \theta} d\theta$  (E) $4 \int_{0}^{\pi/2} \sin^2 \theta d\theta$

13. $\int_{-1}^{1} (1 - |x|) dx$ equals
   (A) 0  (B) $\frac{1}{2}$  (C) 1  (D) 2  (E) none of these

14. $\lim_{x \to \infty} x^{1/2}$
   (A) = 0  (B) = 1  (C) = $e$  (D) = $\infty$  (E) does not exist

15. A differentiable function has the values shown in this table:

   \[
   \begin{array}{|c|c|c|c|c|c|}
   \hline
   x & 2.0 & 2.2 & 2.4 & 2.6 & 2.8 & 3.0 \\
   \hline
   f'(x) & 1.39 & 1.73 & 2.10 & 2.48 & 2.88 & 3.30 \\
   \hline
   \end{array}
   \]

   Estimate $f''(2.1)$.
   (A) 0.34  (B) 0.59  (C) 1.56  (D) 1.70  (E) 1.91
16. If \( A = \int_0^1 e^{-x} \, dx \) is approximated using Riemann sums and the same number of subdivisions, and if \( L, R, \) and \( T \) denote, respectively left, right, and trapezoid sums, then it follows that

(A) \( R \leq A \leq T \leq L \)  \hspace{1cm} (B) \( R \leq T \leq A \leq L \)  \hspace{1cm} (C) \( L \leq T \leq A \leq R \)  \hspace{1cm} (D) \( L \leq A \leq T \leq R \)  \hspace{1cm} (E) None of these is true.

17. If \( \frac{dy}{dx} = y \tan x \) and \( y = 3 \) when \( x = 0 \), then, when \( x = \frac{\pi}{3} \), \( y = \)

(A) \( \ln \sqrt{3} \)  \hspace{1cm} (B) \( \ln 3 \)  \hspace{1cm} (C) \( \frac{3}{2} \)  \hspace{1cm} (D) \( \frac{3\sqrt{3}}{2} \)  \hspace{1cm} (E) 6

18. \( \int_0^6 f(x-1) \, dx = \)

(A) \( \int_{-1}^7 f(x) \, dx \)  \hspace{1cm} (B) \( \int_{-1}^5 f(x) \, dx \)  \hspace{1cm} (C) \( \int_{-1}^5 f(x+1) \, dx \)  \hspace{1cm} (D) \( \int_{-1}^5 f(x) \, dx \)

(E) \( \int_1^7 f(x) \, dx \)

19. The equation of the curve shown below is \( y = \frac{4}{1+x^2} \). What does the area of the shaded region equal?

(A) \( 4 - \frac{\pi}{4} \)  \hspace{1cm} (B) \( 8 - 2\pi \)  \hspace{1cm} (C) \( 8 - \pi \)  \hspace{1cm} (D) \( 8 - \frac{\pi}{2} \)  \hspace{1cm} (E) \( 2\pi - 4 \)

20. Find the slope of the curve \( r = \cos 2\theta \) at \( \theta = \frac{\pi}{6} \).

(A) \( \frac{\sqrt{3}}{7} \)  \hspace{1cm} (B) \( \frac{1}{\sqrt{3}} \)  \hspace{1cm} (C) \( 0 \)  \hspace{1cm} (D) \( \sqrt{3} \)  \hspace{1cm} (E) \( -\sqrt{3} \)

21. A particle moves along a line with velocity, in feet per second, \( v = t^2 - t \). The total distance, in feet, traveled from \( t = 0 \) to \( t = 2 \) equals

(A) \( \frac{1}{3} \)  \hspace{1cm} (B) \( \frac{2}{3} \)  \hspace{1cm} (C) \( 2 \)  \hspace{1cm} (D) \( 1 \)  \hspace{1cm} (E) \( \frac{4}{3} \)
22. The general solution of the differential equation \( \frac{dy}{dx} = \frac{1-2x}{y} \) is a family of

(A) straight lines  (B) circles  (C) hyperbolas
(D) parabolas  (E) ellipses

23. The curve \( x^3 + x \tan y = 27 \) passes through (3,0). Use local linear approximation to estimate the value of \( y \) at \( x = 3.1 \). The value is

(A) -2.7  (B) -0.9  (C) 0  (D) 0.1  (E) 3.0

24. \( \int x \cos x \, dx = \)

(A) \( x \sin x + \cos x + C \)  (B) \( x \sin x - \cos x + C \)
(C) \( \frac{x^2}{2} \sin x + C \)  (D) \( \frac{1}{2} \sin x^2 + C \)  (E) none of these

25. The work done in lifting an object is the product of the weight of the object and the distance it is moved. A cylindrical barrel 2 feet in diameter and 4 feet high is half-full of oil weighing 50 pounds per cubic foot. How much work is done, in foot-pounds, in pumping the oil to the top of the tank?

(A) 100\pi  (B) 200\pi  (C) 300\pi  (D) 400\pi  (E) 1200\pi

26. The coefficient of the \((x - 8)^2\) term in the Taylor polynomial for \( y = x^{2/3} \) centered at \( x = 8 \) is

(A) \( \frac{1}{144} \)  (B) \( \frac{1}{72} \)  (C) \( \frac{1}{9} \)  (D) \( \frac{1}{144} \)  (E) \( \frac{1}{6} \)

27. If \( f'(x) = h(x) \) and \( g(x) = x^2 \), then \( \frac{d}{dx} f(g(x)) = \)

(A) \( h(x^2) \)  (B) \( 3x^2 h(x) \)  (C) \( h'(x) \)  (D) \( 3x^2 h(x^2) \)  (E) \( x^3 h(x^2) \)

28. \( \int_{0}^{\infty} e^{-x^2} \, dx = \)

(A) \(-\infty\)  (B) \(-2\)  (C) 1  (D) 2  (E) \( \infty \)
Part B  TIME: 50 MINUTES

Some questions in this part of the examination require the use of a graphing calculator. There are 17 questions in Part B, for which 50 minutes are allowed. Because there is no deduction for wrong answers, you should answer every question, even if you need to guess.

Directions: Choose the best answer for each question. If the exact numerical value of the correct answer is not listed as a choice, select the choice that is closest to the exact numerical answer.

29. The path of a satellite is given by the parametric equations
   \[ x = 4 \cos t + \cos 12t, \]
   \[ y = 4 \sin t + \sin 12t. \]
   
   The upward velocity at \( t = 1 \) equals
   (A) 2.829  (B) 3.005  (C) 3.073  (D) 3.999  (E) 12.287

30. As a cup of hot chocolate cools, its temperature after \( t \) minutes is given by
   \[ H(t) = 70 + 6e^{-0.4t}. \]
   If its initial temperature was 120°F, what was its average temperature (in °F) during the first 10 minutes?
   (A) 60.9  (B) 82.3  (C) 95.5  (D) 96.1  (E) 99.5

31. An object moving along a line has velocity \( v(t) = t \cos t - \ln(t+2) \), where \( 0 \leq t \leq 10 \).
   The object achieves its maximum speed when \( t = \)
   (A) 3.743  (B) 5.107  (C) 6.419  (D) 7.550  (E) 9.538

32. The graph of \( f' \), which consists of a quarter-circle and two line segments, is shown above. At \( x = 2 \) which of the following statements is true?
   (A) \( f \) is not continuous.
   (B) \( f \) is continuous but not differentiable.
   (C) \( f \) has a relative maximum.
   (D) The graph of \( f \) has a point of inflection.
   (E) none of these
33. Let $H(x) = \int_0^x f(t)dt$, where $f$ is the function whose graph appears below.

![Graph of function f]

The tangent line approximating $H(x)$ near $x = 3$ is $H(x) =$

(A) $-2x + 8$     (B) $2x - 4$     (C) $-2x + 4$     (D) $2x - 8$     (E) $2x - 2$

34. The table shows the speed of an object, in feet per second, at various times during a 12-second interval.

| time (sec) | 0 | 3 | 6 | 7 | 8 | 10 | 12 |
| speed (ft/sec) | 15 | 14 | 11 | 8 | 7 | 3 | 0 |

Estimate the distance the object travels, using the midpoint method with 3 subintervals.

(A) 100 ft     (B) 101 ft     (C) 111 ft     (D) 112 ft     (E) 150 ft

35. In a marathon, when the winner crosses the finish line many runners are still on the course, some quite far behind. If the density of runners $x$ miles from the finish line is given by $R(x) = 20[1 - \cos(1 + 0.03x^2)]$ runners per mile, how many are within 8 miles of the finish line?

(A) 30     (B) 145     (C) 157     (D) 166     (E) 195

36. Find the volume of the solid generated when the region bounded by the y-axis, $y = e^x$, and $y = 2$ is rotated around the y-axis.

(A) 0.296     (B) 0.592     (C) 2.427     (D) 3.998     (E) 27.577

37. If $f(t) = \int_0^t \frac{1}{1+t^2} dt$, then $f'(t)$ equals

(A) $\frac{1}{1+t^2}$     (B) $\frac{2t}{1+t^2}$     (C) $\frac{1}{1+t^4}$     (D) $\frac{2t}{1+t^4}$     (E) $\tan^{-1} t^2$
38. You wish to estimate \( e^t \), over the interval \( |x| < 2 \), with an error less than 0.001. The Lagrange error term suggests that you use a Taylor polynomial at 0 with degree at least
(A) 6   (B) 9   (C) 10   (D) 11   (E) 12

39. Find the volume of the solid formed when one arch of the cycloid defined parametrically by \( x = \theta - \sin \theta \), \( y = 1 - \cos \theta \) is rotated around the x-axis.
(A) 15.708   (B) 17.306   (C) 19.739   (D) 29.609   (E) 49.348

40. Which definite integral represents the length of the first quadrant arc of the curve defined by \( x(t) = e^t \), \( y(t) = 1 - t^2 \)?
(A) \( \int_0^1 \sqrt{1 + \frac{4t^2}{e^{2t}}} \, dt \)   (B) \( \int_{\ln 2}^{\ln 3} \sqrt{1 + \frac{4t^2}{e^{2t}}} \, dt \)   (C) \( \int_0^\infty \sqrt{e^{2t} + 4t^2} \, dt \)
(D) \( \int_0^\infty \sqrt{e^{2t} + 4t^2} \, dt \)   (E) \( \int_{\ln 2}^{\ln 3} \sqrt{e^{2t} + 4t^2} \, dt \)

41. For which function is \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \) the Taylor series about 0?
(A) \( e^x \)   (B) \( e^{-x} \)   (C) \( \sin x \)   (D) \( \cos x \)   (E) \( \ln (1 + x) \)

42. The hypotenuse \( AB \) of a right triangle \( ABC \) is 5 feet, and one leg, \( AC \), is decreasing at the rate of 2 feet per second. The rate, in square feet per second, at which the area is changing when \( AC = 3 \) is
(A) \( \frac{25}{4} \)   (B) \( \frac{7}{4} \)   (C) \( \frac{3}{2} \)   (D) \( \frac{7}{4} \)   (E) \( \frac{7}{2} \)

43. At how many points on the interval \([0, \pi]\) does \( f(x) = 2 \sin x + \sin 4x \) satisfy the Mean Value Theorem?
(A) none   (B) 1   (C) 2   (D) 3   (E) 4

44. Which one of the following series converges?
(A) \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)   (B) \( \sum_{n=1}^{\infty} \frac{1}{n} \)   (C) \( \sum_{n=1}^{\infty} \frac{1}{2n+1} \)

(D) \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \)   (E) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \)

45. The rate at which a purification process can remove contaminants from a tank of water is proportional to the amount of contaminant remaining. If 20% of the contaminant can be removed during the first minute of the process and 98% must be removed to make the water safe, approximately how long will the decontamination process take?
(A) 2 min   (B) 5 min   (C) 18 min   (D) 20 min   (E) 40 min

END OF SECTION I
SECTION II

Part A  TIME: 30 MINUTES  2 PROBLEMS

A graphing calculator is required for some of these problems.  
See instructions on page 4.

1. A function \( f \) is defined on the interval \([0,4]\), and its derivative is  
   \[ f'(x) = e^{\sin x} - 2 \cos 3x. \]
   (a) Sketch \( f' \) in the window \([0,4] \times [-2,5]\).  
      (Note that the following questions refer to \( f \).)
   (b) On what interval is \( f \) increasing?
   (c) At what value(s) of \( x \) does \( f \) have local maxima? Justify your answer.
   (d) How many points of inflection does the graph of \( f \) have? Justify your answer.

2. The rate of sales of a new software product is given by \( S(t) \), where \( S \) is measured in hundreds of units per month and \( t \) is measured in months from the initial release date of January 1, 2012. The software company recorded these sales data:

   \[
   \begin{array}{c|cccccccc}
   t \text{ (months)} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
   S(t) \text{ (100s/month)} & 1.54 & 1.88 & 2.32 & 3.12 & 3.78 & 4.90 & 6.12 \\
   \end{array}
   \]

   (a) Using a trapezoidal approximation, estimate the number of units the company sold during the second quarter (April 1, 2012, through June 30, 2012).
   (b) After looking at these sales figures, a manager suggests that the rate of sales can be modeled by assuming that the rate to be initially 120 units/month and to double every 3 months. Write an equation for \( S \) based on this model.
   (c) Compare the model’s prediction for total second quarter sales with your estimate from part a.
   (d) Use the model to predict the average value of \( S(t) \) for the entire first year. Explain what your answer means.
3. The velocity of an object in motion in the plane for $0 \leq t \leq 1$ is given by the vector \[ v(t) = \left( \frac{1}{\sqrt{4 - t^2}}, \frac{t}{\sqrt{4 - t^2}} \right). \]

(a) When is this object at rest?

(b) If this object was at the origin when $t = 0$, what are its speed and position when $t = 1$?

(c) Find an equation of the curve the object follows, expressing $y$ as a function of $x$.

4. (a) Write the Maclaurin series (including the general term) for $f(x) = \ln(e + x)$.

(b) What is the radius of convergence?

(c) Use the first three terms of that series to write an expression that estimates the value of $\int_0^1 \ln(e + x^2)\,dx$.

5. After pollution-abatement efforts, conservation researchers introduce 100 trout into a small lake. The researchers predict that after $m$ months the rate of growth, $F$, of the trout population will be modeled by the differential equation \[ \frac{dF}{dm} = 0.0002F(600 - F). \]

(a) How large is the trout population when it is growing the fastest?

(b) Solve the differential equation, expressing $F$ as a function of $m$.

(c) How long after the lake was stocked will the population be growing the fastest?

6. (a) A spherical snowball melts so that its surface area shrinks at the constant rate of 10 square centimeters per minute. What is the rate of change of volume when the snowball is 12 centimeters in diameter?

(b) The snowball is packed most densely nearest the center. Suppose that, when it is 12 centimeters in diameter, its density $x$ centimeters from the center is given by $d(x) = \frac{1}{1 + \sqrt{x}}$ grams per cubic centimeter. Set up an integral for the total number of grams (mass) of the snowball then. Do not evaluate.