§P.2: Parent Functions & Transformations

Before you can delve into the wonderfully fascinating world of calculus, you need to be sure you have the tools and skills that will help you survive. Here are some of the more important things that will be crucial to your success.

In calculus, we'll encounter lots and lots of functions. Functions are important relations between two variables, and independent input variable (usually x) and a dependent output variable (usually y or $f(x)$). The thing that differentiates a function from other relations is that **every input can have no more than one output**. An input need not have an output, but if it does, there is only one. This is why the vertical line test works so well. The collection of all allowable inputs is called the **Domain** of the function. The collection of all outputs generated from the inputs is called the **Range**.

Functions are important relations because they are useful in modeling our world. How confusing would it be if we went to the store and independently chose to purchase four delicious Honeycrisp Apples, only to get the register and be told that our bill came to $6 or $12?! Yes! Honeycrisp Apples are expensive and worth every penny, but we would all opt for the lower price. The fact that there should be one price for our items allows us to predict what will happen and establishes a necessary order in the world we live in.

So function cannot have a repeated input. It is legal, though, to have the same output, as long as it comes from different inputs. For instance, the result (output) of the PGA Masters' Champion has gone to Tiger Woods four times, but it has happened in separate years (inputs)—1997, 2001, 2002, and 2005. How bizarre it would be to have co-winners at the Masters Tournament, or, even more bizarre, if Tiger managed to win the Masters twice in one year!

So yes, functions are important, but as you can already tell from the examples above, they can be represented in very different ways.

Throughout the year, we will do our best to describe a functional relation in four ways. This "Rule of Four" will not only provide us with some alternative mathematical "synonyms" to express a result in a different way, but it will help us gain insight into the function and increase our understanding of situation modeled by the function. Abraham Lincoln, when asked by his law partner why he read aloud replied by saying that the more senses he got involved into any effort, the better he was able to understand it. Now I'm not advocating that you sniff and lick your calculus homework, but simply expressing an idea in several ways will help you achieve the level of understanding Lincoln was shooting for, without annoying your classmates.

Here are the four ways we shall represent our functions . . .

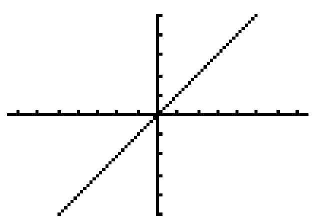
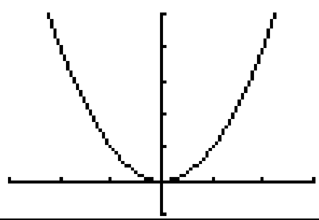
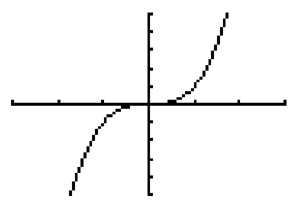
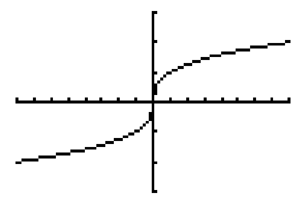
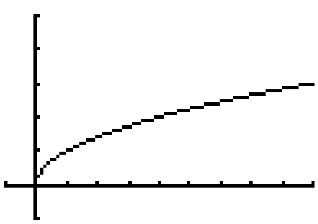
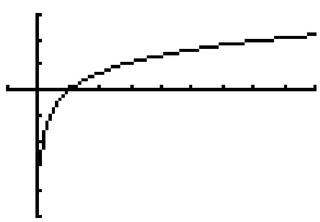
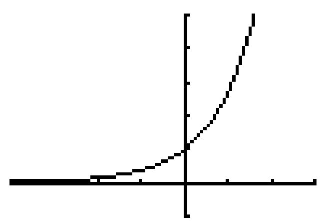
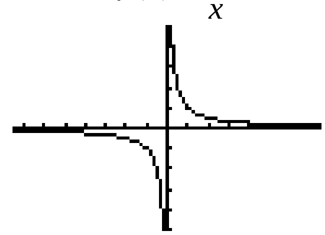
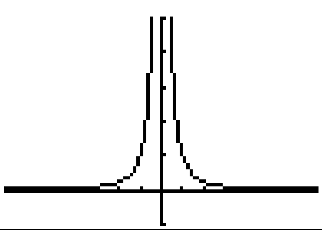
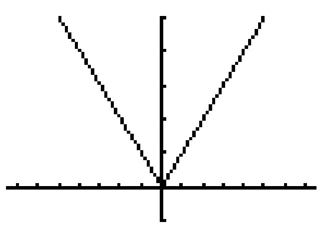
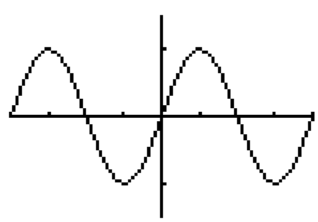
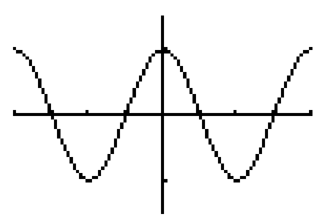
The AP Calculus "Rule of Four"

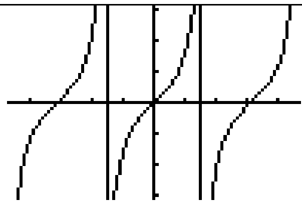
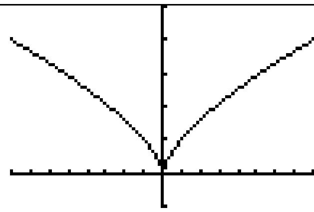
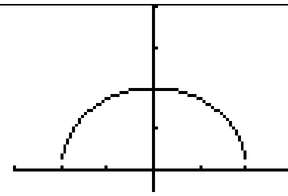
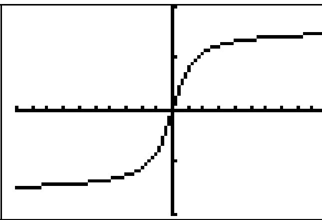
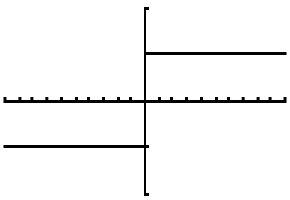
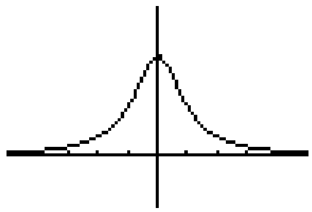
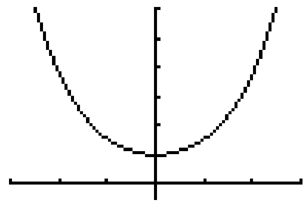
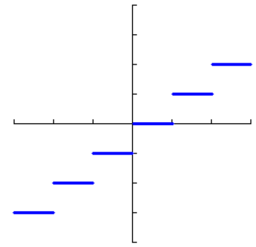
A function may be represented . . .

- **Verbally**, as by a description in words
- **Numerically**, by a set of ordered pairs or a table of values
- **Visually**, by a graph
- **Algebraically**, by an equation

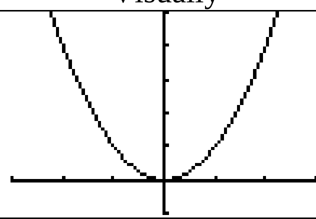
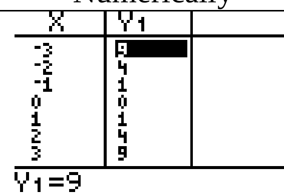
You will encounter many functions this year that will be represented in any of the four ways listed above. Some of these functions will be entirely unfamiliar to you, some of them will have elaborate equations, and some of them will be represented solely by a hideous piece-wise graph, but many, many functions you encounter will be close descendants of a particular matriarch functions.

Here are the algebraic and visual representations of the ones you should know. Be sure to know their domains, ranges, intercepts, symmetry, end behavior (like horizontal asymptotes), and discontinuities (like vertical asymptotes, holes, and jumps).

$f(x)=x$ 	$f(x)=x^2$ 	$f(x)=x^3$ 	$f(x)=x^{1/3}$ 
$f(x)=\sqrt{x}$ 	$f(x)=\ln x$ 	$f(x)=e^x$ 	$f(x)=\dfrac{1}{x}$ 
$f(x)=\dfrac{1}{x^2}$ 	$f(x)=\lvert x\rvert$ 	$f(x)=\sin x$ 	$f(x)=\cos x$ 
$f(x)=\tan x$	$f(x)=x^{2/3}$	$f(x)=\sqrt{r^2-x^2}$	$f(x)=Arc\tan x$

			
$f(x)=\frac{\lvert x\rvert}{x}$ 	$f(x)=\frac{1}{x^2+1}$ 	$f(x)=\frac{1}{2}\left(e^x+e^{-x}\right)=\cosh x$ 	$f(x)=[x]$ 

Let's take one of these functions and express it in the remaining two ways.

Algebraically	Visually	Numerically	Verbally
$f(x)=x^2$			The function is the set of all squares of the inputs.

Of course, some representations of a function are more natural than others. As you already deduced, most of our functions will come to use either algebraically or verbally.

One of the most important skills for AP Calculus success is being able to "see" the graph of a function simply by looking at its equation. Knowing what the graph looks like can help you answer questions about that function quickly and accurately. Knowing a handful of these "mother" functions and how changes in their equations affect their graphs will make life much easier for you.

There are four basic types of transformations: Dilations, Reflections, Shifts, and Absolute Value transformations. We'll start out by concentrating on the first three.

A new function $g(x)$ can be made from an original function $f(x)$. The **standard transformation form** for an original function $f(x)$ is given by the following:

$$g(x)=Af\left(B\left(x-C\right)\right)+D$$

- The two multiplicative constants A and B are dilations and reflections and affect the graphs shape.
- The two additive constants C and D are shifts and change the graphs position.

- The constants outside the function, A and D, affect the y-values exactly as they appear to do.
- The constants inside the function, B and C, affect the x-values in a manner opposite from what they appear to do.

More specifically for each variable:

A: A vertical dilation by a factor of $|A|$.
If $|A|>1$, $f(x)$ has a vertical stretch by a factor of $|A|$.
If $0<|A|<1$, $f(x)$ has a vertical compression by a factor of $1/|A|$
If $A<0$, the graph of $f(x)$ reflects across the *x*-axis (*y*-values interchange).

B: A horizontal dilation by a factor of $|B|$.
If $|B|>1$, $f(x)$ has a horizontal compression by a factor of $|B|$.
If $0<|B|<1$, $f(x)$ has a horizontal stretch by a factor of $1/|B|$.
If $|B|<0$, the graph of $f(x)$ reflects across the *y*-axis (*x*-values interchange)

C: If $C>0$, $f(x)$ shifts/translates *C* units to the right.
If $C<0$, $f(x)$ shifts/translates *C* units to the left.

D: If $D>0$, $f(x)$ shifts/translates *D* units upward.
If $D<0$, $f(x)$ shifts/translates *D* units downward.

When applying a sequence of these transformations, you must apply them in order according to your Dear Aunt Sally's rule, that is, Multiplication before Addition. This means you must apply the Dilations and/or the Reflections first (in any order) and the Shifts/Translations last.

The first step to successfully sketching a transformed graph is to write it in standard transformation form. This will help you identify (in order) both the type and sequence of transformations. Next, you will have to identify the parent function, which will get easier with, guess what, practice. At first, you might have to apply each transformation in order using a light pencil quality for the intermittent graphs until you arrive at the final product. Eventually, though, you should be able to draw it in a single step.

Here's and example . . .

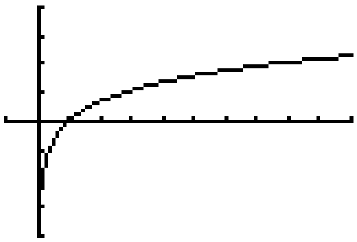
Sketch the function $f(x)=5-7\ln(6+2x)$. State the domain, range, and the equation of any asymptote.

Answer: Hopefully the parent function is obviously $g(x)=\ln x$. Writing in standard transformation form here requires moving the 5 to the back and, more importantly, factoring out the coefficient of *x* from BOTH terms inside the function. Not doing this is the number one mistake students make, which means they have no chance at getting their horizontal shift correct. The final form should look like this:

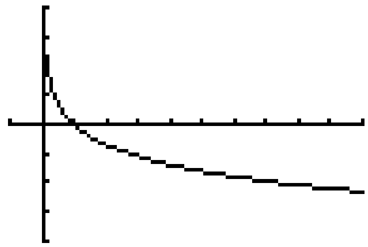
$$f(x)=-7\ln\big(2(x+3)\big)+5$$

Reading the equation from left to right, the graph of $g(x)=\ln x$ will be 1) reflected across the *x*-axis, 2) stretched vertically by a factor of seven, 3) **compressed** horizontally by a factor of two, 4) shifted three units to the **left,** and 5) shifted up five units.

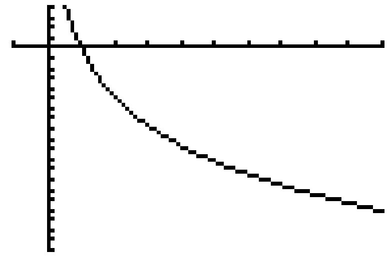
Here's the sequence described above shown one at a time.



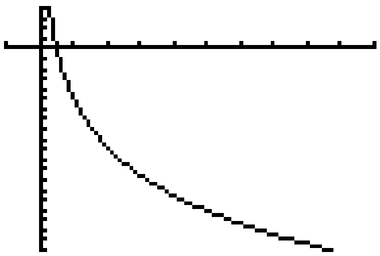
$f(x) = \ln x$



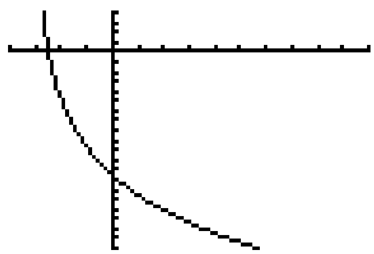
$f(x) = -\ln x$



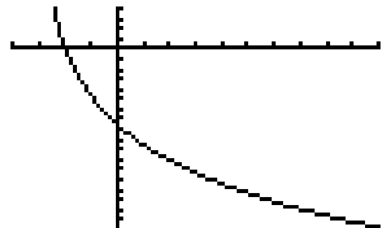
$f(x) = -7\ln x$



$f(x) = -7\ln(2x)$



$f(x) = -7\ln\left(2(x+3)\right)$

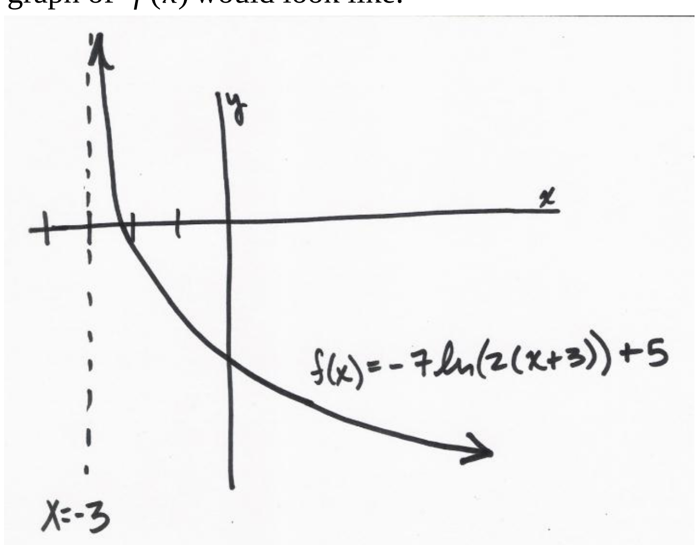


$f(x) = -7\ln\left(2(x+3)\right)+5$

The vertical asymptote originally at $x = 0$ is only affected by the horizontal shift. It is now at $x = -3$. The domain is therefore $D_f : \{x | x > -3\}$ and the range is $R_f : \{y | y \in \mathbb{R}\}$.

Notice that the actual dilations both have the affect of making the graph steeper. For sketching purposes, it is not always necessary to show the actual affect of dilations, but rather the final shape, orientation, and location. For this reason, the truly important transformations are the reflections and shifts. Simply knowing that the graph will be steeper than the original should be adequate, and the intercepts can be found if one desires to have a more accurate sketch.

This knowledge should help you learn to sketch in a single step: finding the final location based on the shifts, then drawing the final shape in its new location with the proper orientation based on any reflections. Here's what my hand-drawn graph of $f(x)$ would look like.



Absolute value transformations are much more exciting simply because they are more uncommon. Most of us have a very intuitive notion for how to take the absolute value of a number—we simply make it positive! This is because the absolute value function is defined to be the non-directed distance to a value from zero. Both -5 and 5 are 5 units away from zero.

Mastering the absolute value transformations is easy if you follow the sequence of evaluating a function. Simply knowing if we're affecting the inputs (*x*-values) or outputs (*y*- or function- values) is all it takes. There are two basic varieties that transform a function $f(x)$.

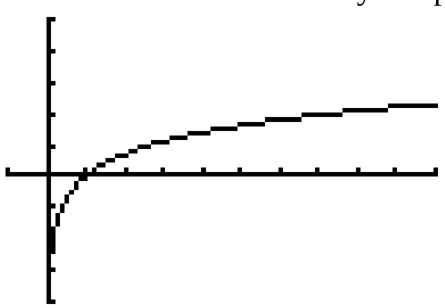
$$g(x) = |f(x)|$$

Notice that the absolute value bars are around the entire function. This means we are only changing the final outputs immediately before plotting the *y*-value. The instructions here tell us to make sure all outputs are non negative. This gives us an easy rule of thumb to follow.

Rule of thumb #1: To obtain the graph of $g(x) = |f(x)|$, leave all the positive *y*-values of $f(x)$ (including $y = 0$) alone, then reflect all negative *y*-value of $f(x)$ across the *x*-axis.

<u>Example:</u> Sketch the graph of $f(x) = |\ln x|$.

Answer: Since the parent function $g(x) = \ln x$ is negative on $0 < x < 1$, we reflect only the *y*-values on this interval across the *x*-axis. The vertical asymptote is still in place, but now affect the graph high up in positive *y*-value land. There is now a very sharp point called a **cusp** at the *x*-intercept $(1, 0)$.



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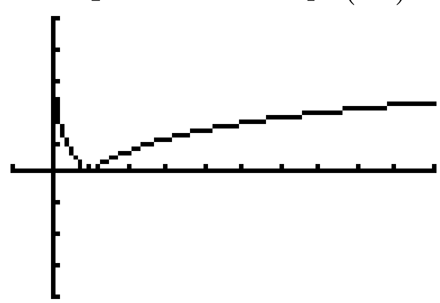


After

*Note: On your calculator, the absolute value button is found under "MATH," "NUM," "ABS." Yes, your calculator has abs (not really). Yes, your calculator has an anti-lock braking system (I wish).

- What does this transformation do to the domain of a function? The range?
- What happens when this transformation is performed correctly on the parent function $f(x) = x$?
- Which of your parent functions are unaffected by this transformation?
- How does Elvis keep making new music?

The second absolute value transformation is my absolute favorite. Let's take a look.

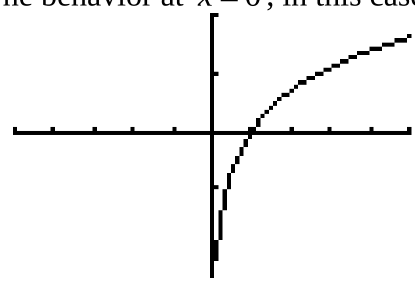
$$g(x) = f(|x|)$$

Notice that the absolute value bars are only around the *x*, ***inside*** the function. This means we are changing the input, prior to applying the rule of the function. This means we are treating ALL inputs as if they were non-negative. Negative *x*-values will graph exactly the same as positive *x*-values. This graph must, therefore, have *y*-axis symmetry. Here's another easy rule of thumb to follow.

Rule of thumb #2: To obtain the graph of $g(x) = f(|x|)$, leave quadrants I and IV of the graph containing all the positive *x*-values (including $x = 0$) alone, then disregarding any part of the original graph in quadrant II and III, reflect quadrants I and IV across the across the *y*-axis.

Example: Sketch the graph of $f(x) = \ln|x|$.

Answer: Since the parent function $g(x) = \ln x$ only lives in positive *x*-land, there is nothing to disregard in quadrants II and III. We simply keep what is already there and generate it's mirror twin on the other side of the *y*-axis. The behavior at $x = 0$, in this case a vertical asymptote, stays the same.



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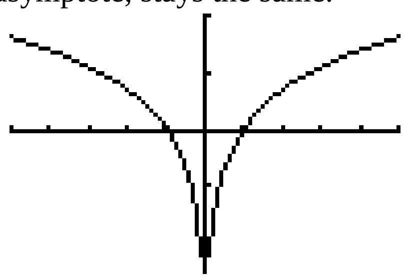

After

- What does this transformation do to the domain of a function? The range?
- What happens when this transformation is performed correctly on the parent function $f(x) = x$?
- Which of your parent functions are unaffected by this transformation?
- Why does everyone always assume "Presley" and not "Costello?"

What rule of thumb can you develop for sketching a function like $g(x) = |f(|x|)|$?

Here's a hint. Just like dilations and reflections can be done in any order because they have the same priority level (multiplication), so too can the absolute value transformations be done in either order.

Try out your rule by sketching $f(x) = |\ln|x||$ then checking with your calculator to see if you're correct.

Sometimes we might need to do a little algebra before we recognize a function as a transformation of one of our parent functions.

Example:

Sketch the function $f(x) = \dfrac{x-1}{x}$ without a calculator, then find the domain and range.

This function appears to be close to our reciprocal parent function $y = \dfrac{1}{x}$, but not quite. Whenever we have a rational expression where the degree of the numerator is greater than or equal to the degree of the denominator, we can use **long division** to rewrite the expression.

In general, a fraction can rewritten as

$$\frac{dividend}{divisor} = quotient + \frac{remainder}{divisor}$$

$$\begin{array}{r} 1 \\ x\overline{)\,x-1} \\ \underline{-x} \\ -1 \end{array}$$

So, $\dfrac{x-1}{x} = 1 - \dfrac{1}{x}$

and $f(x) = -\dfrac{1}{x} + 1$

$$\boxed{\begin{array}{l} D_f : \{x \mid x \neq 0\} \\ R_f : \{y \mid y \neq 1\} \end{array}}$$

- Notice how the parent function has origin symmetry but this new transformed graph does not. Simply shifting the graph vertically "ruins" the origin symmetry. Can you verify this algebraically?