§P.6—Fun with Functions

Now that we know how to find domains and ranges of functions, as well as being able to graph them, we will combine these skills with our skills of simplifying expressions to work with and answer questions about other types of functions. Many of the functions we will work with throughout calculus will belong to the polynomial family of functions. Much time was spent in Precalculus studying this very important family, but here's a quick review.

Definition:

$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$ is a **polynomial function** of *x* if $a_i \in \mathbb{R}$, $a_n \neq 0$, and $n \in \{0,1,2,3,\ldots\}$.

Is it all coming back to you? Relax. The definition above just means that the exponents of the function have to be non-negative integers, like 0, 1, 2, 3, . . . The *a* values are the coefficients of the respective terms. a_n is called the **leading coefficient** because it's associated with the largest exponent, *n*, called the **degree** of the polynomial function, and a_0 is the constant.

Polynomial functions have many important properties summarized below:

- Domain of all real numbers
- Continuous for all real *x*
- Have end behavior, meaning they either increase or decrease without bound as *x* approaches negative and positive infinity.
- Have local behavior consisting of both *x*- and *y*-intercepts, turning points (relative extrema), and inflection points (where the curvature/concavity changes).
- *x*-intercepts are also called zeros, roots, or solutions to $f(x)=0$
- If *n* is even, $f(x)$ has
  - Minimum of zero roots (doesn't have to cross the *x*-axis)
  - Maximum of n roots
  - An odd number of relative extrema up to $n-1$.
  - Either an upper or lower bound
  - Same end behavior (enters and exits the top or enters and exits the bottom)
- If *n* is odd, $f(x)$ has
  - Minimum of one root (must cross the *x*-axis at least once)
  - Maximum of n roots
  - An even number of relative extrema up to $n-1$.
  - Has a range of all real numbers (bounded neither above nor below)
  - Opposite end behavior (enters top and exits bottom or enters the bottom and exits the top)
- Have inflection points where the curvature/concavity changes (for $n \geq 3$)
- Can be named according to how many terms they have
  - One term: Monomial
  - Two terms: Binomial
  - Three terms: Trinomial
  - Four or more terms: Polynomial
- Can be named according to the degree, *n*, of the polynomial
  - Degree zero: Constant
  - Degree one: Linear
  - Degree two: Quadratic
  - Degree three: Cubic
  - Degree four: Quartic
  - Degree five: Quintic
  - Degree *n*: *n*th degree polynomial
- If $a_n > 0$, $f(x)$ exists the top. If $a_n < 0$, $f(x)$ exists the bottom. That is, the sign of the leading coefficient will always give you the right end-behavior

Knowing all the information above, combined with some information from the sign of the leading coefficient will allow you to quickly gather great insight into a polynomial of any degree of any number of terms.

Example 1:
Discuss the properties of $f(x) = 8x^4 + 50x^3 - 128x^2 - 2x^5 - 216x + 288$

Knowing a few theorems will allow us to get even farther.

Remainder Theorem
If $f(x)$ is divided by $x - c$ to give a quotient of $g(x)$ with a remainder of r, then $f(c) = r$ and $g(x)$ has degree $n-1$. $f(x)$ can then be written equivalently as $f(x) = (x-c)\cdot g(x) + r$ or $\dfrac{f(x)}{x-c} = g(x) + \dfrac{r}{x-c}$.

We can divide polynomials in much the same way we can divide numbers, using long division. When we are dividing by a linear factor, we can also use synthetic division.

Example 2:
Divide $f(x) = -2x^5 + 8x^4 + 50x^3 - 128x^2 - 216x + 288$ by $x - 2$. Rewrite $f(x)$ as a product of $x - 2$ and the quotient plus the remainder.

Example 3:
Determine if $x = -2$ is a root of $f(x) = x^3 - 7x - 6$ by using synthetic substitution and direct substitution.

Another important property of functions, including polynomial functions, is **monotonicity**. We draw and read graphs like we read a book . . . from cover to cover . . . No, seriously, from left to right. As we follow the path of the graph of a particular function over any **OPEN INTERVAL**, it can either rise, fall, or stay steady (horizontal). A function that ONLY exhibits one of these three traits over its entire domain is called a **monotonic function**.

- A function that only rises from left to right on any and all open intervals in its domain is said to be **monotonic increasing**.
- A function that only falls from left to right on any and all open intervals in its domain is said to be **monotonic decreasing**.
- A function that neither rises nor falls from left to right on any and all open intervals in its domain is said to be (monotonic) **constant**.

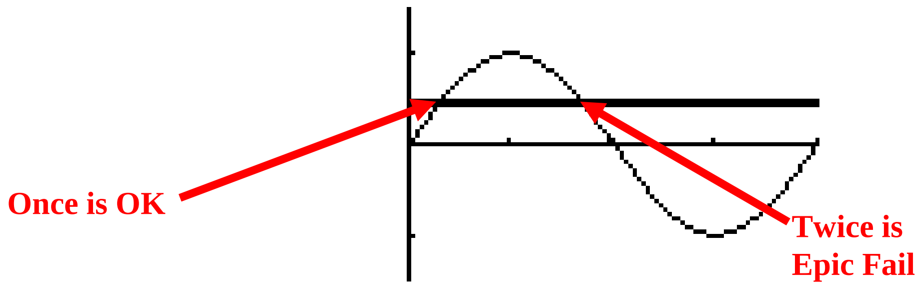
For now, monotonicity is something that will examined graphically. With a few calculus skills, we'll be able to quantifiably determine intervals of monotonicity.

Parent functions such as $f(x) = x$, $f(x) = x^3$, and $f(x) = \tan x$ are all monotonic increasing.
*Functions that are not monotonic increasing/decreasing over their entire domain can still be so on subintervals of their domain.

Example 4:
Determine the largest open intervals on which the function $f(x) = \sin x$ is monotonic decreasing, $0 \le x \le 2\pi$.

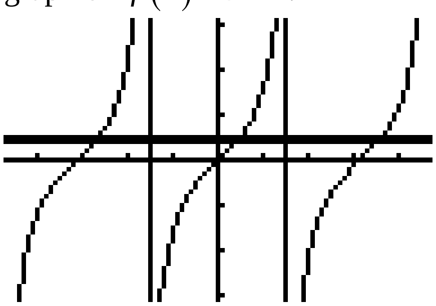
Another important characteristic of special functions is **one-to-oneness**. To be a function, each input can have no more than one output. It is perfectly OK, though, for different inputs to go to the same output (for example $\sin 0 = \sin \pi = \sin 2\pi = 0$). A function that is **one-to-one** is special because each output is associated with only one input. Think of it as each coordinate on the graph being in a mutually monogamous, romantic relationship. For each *x* there's only one *y*, and for each *y*, there's only one *x*.

If you recall, the quick graphical check for one-to-oneness is the horizontal line test. The graph of $f(x) = \sin x$ is NOT one-to-one, as already mentioned, and it clearly fails the horizontal line test.



All one-to-one functions are monotonic. Parent functions such as $f(x) = x$, $f(x) = x^3$, $f(x) = e^x$, and $f(x) = \ln x$ are all one-to-one and, therefore, monotonic. A more interesting question is "Are all monotonic functions one-to-one?" Can you think of a monotonic parent function (or ANY function) that **fails** the horizontal line test? How about $f(x) = \tan x$?

It is monotonic increasing since it rises from left to right over its entire domain (everywhere except of odd pi halves where the vertical asymptotes are), but it fails the horizontal line test as miserably and often as the graph of $f(x) = \sin x$.



- Just as we can look at specified intervals to discuss monotonicity, so can we talk about one-to-oneness on specified intervals.
- $f(x) = \tan x$ is one-to-one on the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, between vertical asymptotes.

Although monotonicity is an interesting characteristic of functions, like a type of haircut is on a person, the more important, or consequential characteristic of the two is one-to-oneness, which I guess would be the natural hair color of a person. The reason has to do with the **necessary condition** in which a function is considered invertible.

A function $f(x)$ that is one-to-one will posses an **inverse function** $g(x) = f^{-1}(x)$ (read as "*f* inverse of *x*"), such that $f(g(x)) = x = g(f(x))$.

This means that any and every one-to-one function will have an associated function that will "undo" the work of the original function. Like any good method for coding secret messages will have a unique way to decode the message, so does an inverse function restore the output of a function to the original input of that function.

Finding these inverse functions is sometimes intuitive, sometimes algebraically available, and sometimes very, very difficult to find.

Example 5:
Verify that $f(x)=2x+3$ is one-to-one, then find its inverse function $f^{-1}(x)$. What do you notice about the slopes of the inverse functions?

Sometimes finding the inverse of a function requires a bit of algebraic finesse. Once we have the inverse relation, we can analyze its domain to determine if it is, in fact, a function.

Example 6:
Find the inverse function of $f(x)=\dfrac{x-3}{x+4}$. What do you notice about the vertical and horizontal asymptotes of the two functions? The intercepts? Domains/Ranges?