§1.2—Properties of Limits

When working with limits, you should become adroit and adept at using limits of generic functions to find new limits of new functions created from combinations and modifications to those generic functions. I'll show you what I mean, but first, some important properties of limits that make it all work.

Properties of Limits

1. (addition/subtraction) $\lim_{x\to c}\left[f(x)\pm g(x)\right]=\lim_{x\to c}f(x)\pm\lim_{x\to c}g(x)$
2. (constant multiple) $\lim_{x\to c}k\cdot f(x)=k\lim_{x\to c}f(x)$
3. (multiplication) $\lim_{x\to c}\left[f(x)\cdot g(x)\right]=\lim_{x\to c}f(x)\cdot\lim_{x\to c}g(x)$
4. (division) $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}$, $\lim_{x\to c}g(x)\neq 0$
5. (exponentiation) $\lim_{x\to c}\left[f(x)\right]^n=\left[\lim_{x\to c}f(x)\right]^n$
6. (composition) If $\lim_{x\to c}g(x)=L$, then $\lim_{x\to c}f(g(x))=\lim_{x\to L}f(x)$

In general, the limit can be taken of each piece including a variable in any expression independently, then, the results of these limits may be combined using the algebraic rules of the expression, for example:

$$\lim_{x\to c}\frac{\sqrt{3x+f(x)}}{x\cdot g^2(x)}=\frac{\sqrt{3\lim_{x\to c}x+\lim_{x\to c}f(x)}}{\lim_{x\to c}x\cdot\left(\lim_{x\to c}g(x)\right)^2}$$

Example 1:

Given $\lim_{x\to3} f(x)=8$, $\lim_{x\to3} g(x)=-2$, and $\lim_{x\to3} h(x)=0$, find the limits that exist.

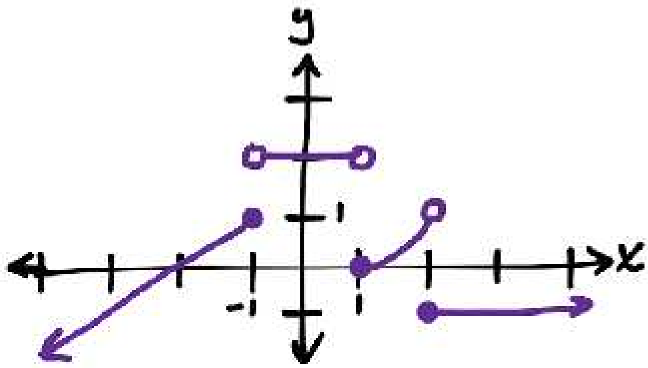
(a) $\lim_{x\to3}\left[2f(x)-4g(x)\right]=$

(b) $\lim_{x\to3}\left[2g(x)\right]^2=$

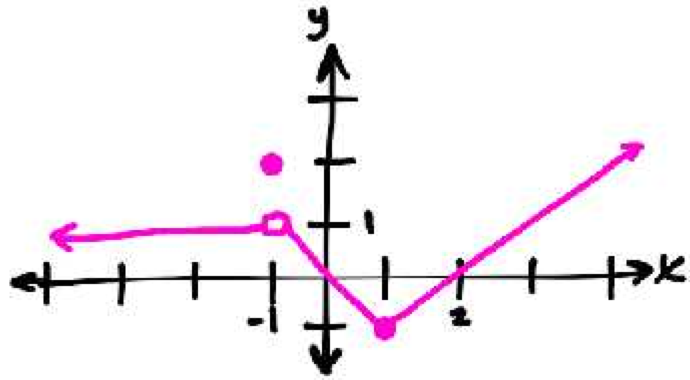
(c) $\lim_{x\to3}\left(\frac{\sqrt[3]{f(x)}}{g(x)}+\frac{4h(x)}{x+7}\right)=$

Example 2:

Given the graphs of f and g are given below.



graph of f



graph of g

Determine whether the following limits exist. If they do, then find the limit.

(a) $\lim_{x\to0}\left[2[f(x)]^2+3g(x)=\right]$

(b) $\lim_{x\to-3}\frac{x^2g(x)}{f(x)}=$

(c) $\lim_{x\to-3} g(f(x))=$

(d) $\lim_{x\to-1} g\left(x^2\right)=$

(e) $\lim_{x\to1^-}\frac{2f(x)}{[g(x)]^2}=$

(f) $\lim_{x\to-1}\left(5-\sqrt{x^2\left(3+f(x)\right)}\right)=$

The Squeeze Theorem

If $g(x) \le f(x) \le h(x)$ for all *x*, except possibly at $x = c$, and **if** $\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x)$,

then $\lim_{x \to c} f(x) = L$

If you have a function that is everywhere contained between two other functions, then at the point where the two outer functions are sandwiched or squeezed together through a single point, then the one in between them **must** pass through that point as well. Think of it as thousands of concert goers from all over the stadium leaving the concert at the end of the night through a **single** turnstile.

Example 3:

Using your calculator, graph $y_1 = x^2$, $y_2 = -x^2$, and $y_3 = x^2 \sin\left(\frac{1}{x}\right)$.

X-window: $[-0.1, 0.1]$ Y-window: $[-0.005, 0.005]$. Use the graphs to determine $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

The theorem itself is easy enough to understand. It's the formulating of your argument that requires a little training.

Example 4:

If $-3x \le g(x) \le x^2 + x + 3$, find $\lim_{x \to -1} g(x)$.