§1.3—Limits at Infinity

When we take the limit of a function at infinity, we are interested in the **end-behavior** of a graph. We can write the analysis of each end-behavior of a function $f(x)$ using the following notations:

$$\lim_{x\to\infty} f(x) \text{ or } \lim_{x\to-\infty} f(x)$$

Assuming a function exists on either end, for infinitely large and infinitely small values of *x*, it is limited in what it can do. We will now analyze these possibilities.

Example 1:
Evaluate the following limits without a calculator:

(a) $\lim\limits_{x\to\infty}\left(3x^5-4x^3+2x^2-7x+11\right)=$ (b) $\lim\limits_{x\to-\infty}\left(3x^5-4x^3+2x^2-7x+11\right)=$ (c) $\lim\limits_{x\to\infty} e^{-x}=$

(d) $\lim\limits_{x\to-\infty} e^{-x}=$ (e) $\lim\limits_{x\to\infty}\sin x=$ (f) $\lim\limits_{x\to-\infty} x\sin x=$ (g) $\lim\limits_{x\to\infty}\frac{1}{x}\sin x=$

The last case is of particular interest to all math teachers, most math students, and very few chimpanzees. Here's why:

Definition:

$$\lim_{x\to\infty} f(x)=L \textbf{ or } \lim_{x\to-\infty} f(x)=L$$

$$\Leftrightarrow$$

$f(x)$ has a **Horizontal Asymptote** (HA) at $y=L$

Horizontal Asymptotes are NOT, repeat, NOT discontinuities. A graph can cross its HA any number of times (although they don't have to). The HA just tells us the *y*-values the graph approaches.

Example 2:
Find the equations of any horizontal asymptotes of $f(x) = \arctan x$ by analyzing the end-behaviors of the graph. Use proper notation.

Many rational functions have horizontal asymptotes.

Example 3:
Evaluate the end behavior of the following rational functions. Use correct notation.

(a) $f(x) = \dfrac{3x^2 - 8x + 12}{5x^3 + 4x^2 - x - 2}$ (b) $g(x) = \dfrac{3x^5 - 8x^3 + 12}{5x^5 + 4x^2 - x - 2}$ (c) $h(x) = \dfrac{4x^3 - 2x^2 + 4x - 7}{-2x^2 + 5x - 9}$

Example 3c above did not have a horizontal asymptote, but it did have another type of end-behavior asymptote—a **Slant Asymptote** (SA). If the degree of the numerator of a rational function is greater than the denominator's degree, the **end-behavior model** will be a polynomial whose degree is the difference of the degrees of the numerator and denominator. The end behavior model will have the same end behaviors as the original function, and it will tell us *how* the graph of the function goes to

Example 4:
Find the equation of the end-behavior asymptote for $h(x) = \dfrac{4x^3 - 2x^2 + 4x - 7}{-2x^2 + 5x - 9}$

Sometimes, the degrees of the numerator and denominator are not as explicit. This often occurs when either the numerator or denominator are under a **radical**. For these types of problems, we can use a similar analysis. Functions like these may have two or one end-behavior asymptotes. **It's important to analyze the effective growth rates to get the value of the limit, then, based upon the type of infinity we are approaching, determine the sign.**

Example 5:
Using correct notation, determine the end-behaviors of the following functions. Write the equation(s) of any end-behavior asymptotes.

(a) $f(x)=\dfrac{4x-2}{\sqrt{9x^2+8x+1}}$ (b) $g(x)=\dfrac{-2x^3-5}{\sqrt{11x^6+3x^2+1}}$ (c) $K(x)=\dfrac{-5x^2-4}{\sqrt{11x^6+2x^2+8}}$

<u>Limits at infinity: Bonus Problems!</u>
Different functions grow at different rates for large values: log functions grow slowly, polynomials grow faster by order of degree, exponential growth functions grow faster than a polynomial of any degree, factorials, like $4!=4\cdot3\cdot2\cdot1$, are next, and the king of growth is $f(x)=x^x$. If you have a ratio of these types of functions, you can find limits at infinity by analyzing respective growth rates.

List of functions by how fast they grow for large *x* in increasing order

$$k\to\log x\to x^n\to b^x\to x!\to x^x$$

Note 1: k is a constant
Note 2: for $x!$, x is a Whole number: $0,1,2,3,\dots$
Note 3: there is no Note 4

Example 6:
Evaluate the following by analyzing the comparative growth rate of the numerators and denominators.

(a) $\lim_{x\to\infty} \dfrac{x^3}{1.01^x} =$

(b) $\lim_{x\to\infty} \dfrac{99999999}{5x - \cos x^2} =$

(c) $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt[9]{x}} =$

(d) $\lim_{x\to\infty} \dfrac{100^x}{x^x} =$

(e) $\lim_{x\to\infty} \dfrac{x!}{x^x} =$

(f) $\lim_{x\to\infty} \dfrac{x^x}{x!} =$

Example 7:
Evaluate each of the following.

(a) $\lim_{x\to\infty} \dfrac{8-3^x}{4+3^x} =$

(b) $\lim_{x\to-\infty} \dfrac{8-3^x}{4+3^x} =$