

§1.4—Algebraic Limits

Finding limits without a graph or a calculator is where the fun is to be had. Often, this requires some traditional algebraic manipulations. We will look at a few of these methods in this section; some will be a quick review, some will be new. The names of the methods are not important, but recognizing the situations in which each method arises **IS** important.

Example 1: Direct Substitution

Always try 1st. If you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the goal is to manipulate the expression algebraically to eradicate the “bad guys” from the numerator or denominator that cause the $\frac{0}{0}$ or $\frac{\infty}{\infty}$. After the “bad guys” are gone, direct substitution will work.

$$(a) \lim_{x \rightarrow \frac{5\pi}{3}} \csc x =$$

$$(b) \lim_{x \rightarrow -4^-} \sqrt{-x} =$$

Example 2: Knowing a VA exists

Remember any time direct substitution yields $\frac{\neq 0}{0}$, there is a VA at that x -value

$$(a) \lim_{x \rightarrow 1^+} \frac{x^3 + 1}{x - 1} =$$

$$(b) \lim_{x \rightarrow \pi^+} \frac{x}{\sin x} =$$

Example 3: Limits at Infinity

These are easy to spot.

$$(a) \lim_{x \rightarrow \infty} \frac{5x^2 - 2x + 3}{-4x^2 + 1} =$$

$$(b) \lim_{x \rightarrow -\infty} \ln|x| =$$

Example 4: Piecewise Functions

These, too, are easy to spot.

$$(a) \text{ If } f(x) = \begin{cases} x^2 - 3, & x \leq 2 \\ \frac{1}{2}x + 1, & x > 2 \end{cases}, \quad \lim_{x \rightarrow 2} f(x) =$$

$$(b) \text{ If } f(x) = \begin{cases} \sin x, & x \neq \pi \\ 1, & x = \pi \end{cases}, \quad \lim_{x \rightarrow \pi} f(x) =$$

Example 5: The Squeeze Theorem

Spotting this one is like spotting a Bwana on a Safari.

If $\ln x + x^2 \leq J(x) \leq e^{x-1}$ for all x in an interval containing $x = 1$, except possibly at $x = 1$ itself, find $\lim_{x \rightarrow 1} J(x)$. Justify.

The rest of the methods below deal with the algebraic methods used to eradicate the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ from the expression so that direct substitution may be used.

Example 6: F.A.D.O. (Factor and Divide out)

We've already seen this method. It works well in the $\frac{0}{0}$ case with rational functions.

$$(a) \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} =$$

$$(b) \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} =$$

$$(c) \lim_{x \rightarrow -1} \frac{x^4 - x^3 - x^2 - 3x - 4}{2x^2 - x - 3} =$$

Example 7: RATCON (short for RAtionalization CONjugation)

This works well in the $\frac{0}{0}$ case when you have two terms either in the numerator or denominator, where at least one term is a radical.

$$(a) \lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} =$$

$$(b) \lim_{x \rightarrow 3} \frac{x - 3}{\sqrt{x+2} - \sqrt{5}} =$$

Example 8: LCM/LCM

This method works well in the $\frac{0}{0}$ case when there is a complex or compound fraction.

$$(a) \lim_{x \rightarrow 0} \frac{\left[\frac{1}{2+x} \right] - (1/2)}{x} =$$

$$(b) \lim_{x \rightarrow 0} \frac{x}{\frac{1}{6} + \frac{1}{x-6}} =$$

Example 9: Sledgehammer

This method works well in the $\frac{0}{0}$ case when there is obvious math to do, like expanding or distributing factors.

$$(a) \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} =$$

$$(b) \lim_{h \rightarrow 0} \frac{(x+h)^2 - 2(x+h) + 1 - (x^2 - 2x + 1)}{h} =$$

We will now meet a new parent function: **Señor Salta**

Example 10:

Sketch the graph of $f(x) = \frac{|x|}{x} = \frac{x}{|x|}$ by decomposing the function into a piecewise function using the definition of $|x|$. Then evaluate the limits below.

$$(a) \lim_{x \rightarrow 0^-} \frac{|x|}{x} =$$

$$(b) \lim_{x \rightarrow 0^+} \frac{|x|}{x} =$$

$$(c) \lim_{x \rightarrow 0} \frac{|x|}{x} =$$

$$(d) \lim_{x \rightarrow 3} \frac{|x|}{x} =$$

Example 11:

Evaluate the following:

(a) $\lim_{x \rightarrow 5^-} \frac{|2x-10|}{x-5}$

(b) $\lim_{x \rightarrow 5^+} \frac{|2x-10|}{x-5}$

(c) $\lim_{x \rightarrow 5} \frac{|2x-10|}{x-5}$

Example 12:

Evaluate the following:

(a) $\lim_{x \rightarrow -3^-} \frac{x^2|2x+6|}{4x+12}$

(b) $\lim_{x \rightarrow 6^+} \frac{x^2-8x+12}{|2x-12|}$

(c) $\lim_{x \rightarrow 7} \frac{(\sin x)|3x-21|}{x^2-49}$

Very often, we must evaluate limits involving trig functions, and very often, these limits are not easy to evaluate unless we have one of three things: (1) a calculator, (2) a smart, helpful friend, or (3) a couple of trig limits memorized.

Example 13:

With a calculator, evaluate the following, then memorize the results:

(a) $\lim_{x \rightarrow 0} \frac{\sin x}{x} =$

(b) $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} =$

Now try

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} =$$

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{5x} =$$

$$\lim_{x \rightarrow 0} \frac{\sin(x/2)}{x/2} =$$

$$\lim_{x \rightarrow 4} \frac{x-4}{\sin(x-4)} =$$

Now try

$$\lim_{x \rightarrow 0} \frac{x}{1 - \cos x} =$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos \pi x}{\pi x} =$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos(x/9)}{x/9} =$$

$$\lim_{x \rightarrow -7^-} \frac{1 - \cos(x+7)}{x+7} =$$

We can now use these memorized limits in conjunction with our limit properties. But WHAT if our expressions are slightly different??

Example 14:

(a) $\lim_{x \rightarrow 0} \frac{\sin 7x}{6x} =$

(b) $\lim_{x \rightarrow 0} \frac{3x}{\sin \pi x} =$

(c) $\lim_{x \rightarrow 0} \frac{4 - 4 \cos 11x}{44x} =$

Example 15:

Evaluate the following.

(a) $\lim_{x \rightarrow 0} \frac{x + \sin x}{x}$

(b) $\lim_{x \rightarrow 0} \frac{x}{\tan x}$

(c) $\lim_{\theta \rightarrow 0} \frac{\sec \theta - 1}{\theta \sec \theta}$

(d) $\lim_{x \rightarrow 0} \frac{4 \sin 5x}{x \cos 2x}$

(e) $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 4x}$

(f) $\lim_{x \rightarrow 0} \frac{\cot 11x}{\csc 25x}$

(g) $\lim_{x \rightarrow 0} \frac{\cos 14x}{\csc 9x}$

(h) $\lim_{x \rightarrow 0} \frac{\csc 7x}{\sin \pi x}$

(i) $\lim_{t \rightarrow 0} \frac{\sin^2 2t}{3t^2} =$

(j) $\lim_{x \rightarrow 0} \frac{\sin^2 2x}{\sin^2 5x} =$

(k) $\lim_{x \rightarrow 0} \frac{e^x \cos^2 8x}{\sec 4x} =$

(l) $\lim_{x \rightarrow 4^+} \frac{\tan(x-4) + |2x-8|}{x^2 - 4} =$

Useful Trig Limits

$$\lim_{x \rightarrow 0} \frac{\sin mx}{mx} = \lim_{x \rightarrow 0} \frac{mx}{\sin mx} = \lim_{x \rightarrow 0} \frac{mx}{\tan mx} = \lim_{x \rightarrow 0} \frac{\tan mx}{mx} = 1$$

$$\lim_{x \rightarrow 0} \frac{\sin mx}{nx} = \lim_{x \rightarrow 0} \frac{mx}{\sin nx} = \lim_{x \rightarrow 0} \frac{mx}{\tan nx} = \lim_{x \rightarrow 0} \frac{\tan nx}{nx} = \frac{m}{n}$$

$$\lim_{x \rightarrow 0} \frac{\sin mx}{\sin nx} = \lim_{x \rightarrow 0} \frac{\tan mx}{\tan nx} = \lim_{x \rightarrow 0} \frac{\tan mx}{\sin nx} = \lim_{x \rightarrow 0} \frac{\sin mx}{\tan nx} = \frac{m}{n}$$

$$\lim_{x \rightarrow 0} \frac{\csc mx}{\csc nx} = \lim_{x \rightarrow 0} \frac{\cot mx}{\cot nx} = \lim_{x \rightarrow 0} \frac{\cot mx}{\csc nx} = \lim_{x \rightarrow 0} \frac{\csc mx}{\cot nx} = \frac{n}{m}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos mx}{mx} = \lim_{x \rightarrow 0} \frac{\cos mx - 1}{mx} = 0$$

Example 16: General Cleverness

This is a great method when all else fails and you have to make a dugout canoe from a toothpick.

$$(a) \lim_{x \rightarrow 0} \frac{5x^2}{1 - \cos x} =$$

$$(b) \lim_{x \rightarrow 5} \frac{3x - 15}{\sqrt{x^2 - 10x + 25}} =$$

$$(c) \lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\sin x - \cos x} =$$

$$(d) \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} =$$

$$(e) \lim_{x \rightarrow \pi/3} \frac{12 \cos^2 x + 2 \cos x - 4}{4 \cos x - 2} =$$