§1.4—Algebraic Limits

Finding limits without a graph or a calculator is where the fun is to be had. Often, this requires some traditional algebraic manipulations. We will look at a few of these methods in this section; some will be a quick review, some will be new. The names of the methods are not important, but recognizing the situations in which each method arises **IS** important.

Example 1: Direct Substitution

Always try 1st. If you get $\frac{0}{0}$ or $\frac{\infty}{\infty}$, the goal is to manipulate the expression algebraically to eradicate the "bad guys" from the numerator or denominator that cause the $\frac{0}{0}$ or $\frac{\infty}{\infty}$. After the "bad guys" are gone, direct substitution will work.

(a) $\lim\limits_{x \to \frac{5\pi}{3}} \csc x =$

(b) $\lim\limits_{x \to -4^-} \sqrt{-x} =$

Example 2: Knowing a VA exists

Remember any time direct substitution yields $\frac{\neq 0}{0}$, there is a VA at that x-value

(a) $\lim\limits_{x \to 1^+} \frac{x^3+1}{x-1} =$

(b) $\lim\limits_{x \to \pi^+} \frac{x}{\sin x} =$

Example 3: Limits at Infinity

These are easy to spot.

(a) $\lim\limits_{x \to \infty} \frac{5x^2-2x+3}{-4x^2+1} =$

(b) $\lim\limits_{x \to -\infty} \ln|x| =$

Example 4: Piecewise Functions

These, too, are easy to spot.

(a) If $f(x) = \begin{cases} x^2-3, & x \le 2 \\ \frac{1}{2}x+1, & x > 2 \end{cases}$, $\lim\limits_{x \to 2} f(x) =$

(b) If $f(x) = \begin{cases} \sin x, & x \neq \pi \\ 1, & x = \pi \end{cases}$, $\lim\limits_{x \to \pi} f(x) =$

Example 5: <u>The Squeeze Theorem</u>
Spotting this one is like spotting a Bwana on a Safari.

If $\ln x + x^2 \le J(x) \le e^{x-1}$ for all x in an interval containing $x=1$, except possibly at $x=1$ itself, find $\lim_{x\to 1} J(x)$. Justify.

The rest of the methods below deal with the algebraic methods used to eradicate the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ from the expression so that direct substitution may be used.

Example 6: <u>F.A.D.O. (Factor and Divide out)</u>
We've already seen this method. It works well in the $\frac{0}{0}$ case with rational functions.

(a) $\lim_{x\to 2}\frac{x^2-4}{x-2}=$ (b) $\lim_{x\to 1}\frac{x^3-1}{x-1}=$ (c) $\lim_{x\to -1}\frac{x^4-x^3-x^2-3x-4}{2x^2-x-3}=$

Example 7: <u>RATCON (short for RATionalization CONjugation)</u>
This works well in the $\frac{0}{0}$ case when you have two terms either in the numerator or denominator, where at least one term is a radical.

(a) $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}=$ (b) $\lim_{x\to 3}\frac{x-3}{\sqrt{x+2}-\sqrt{5}}=$

Example 8: <u>LCM/LCM</u>

This method works well in the $\frac{0}{0}$ case when there is a complex or compound fraction.

(a) $\lim_{x\to 0} \frac{\left[1/(2+x)\right]-(1/2)}{x} =$

(b) $\lim_{x\to 0} \frac{x}{\frac{1}{6}+\frac{1}{x-6}} =$

Example 9: <u>Sledgehammer</u>

This method works well in the $\frac{0}{0}$ case when there is obvious math to do, like expanding or distributing factors.

(a) $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} =$

(b) $\lim_{h\to 0} \frac{(x+h)^2-2(x+h)+1-\left(x^2-2x+1\right)}{h} =$

We will now meet a new parent function: **Señor Salta**

Example 10:

Sketch the graph of $f(x)=\frac{|x|}{x}=\frac{x}{|x|}$ by decomposing the function into a piecewise function using the definition of $|x|$. Then evaluate the limits below.

(a) $\lim_{x\to 0^-} \frac{|x|}{x} =$

(b) $\lim_{x\to 0^+} \frac{|x|}{x} =$

(c) $\lim_{x\to 0} \frac{|x|}{x} =$

(d) $\lim_{x\to 3} \frac{|x|}{x} =$

Example 11:
Evaluate the following:

(a) $\lim\limits_{x\to 5^-} \dfrac{|2x-10|}{x-5}$ (b) $\lim\limits_{x\to 5^+} \dfrac{|2x-10|}{x-5}$ (c) $\lim\limits_{x\to 5} \dfrac{|2x-10|}{x-5}$

Example 12:
Evaluate the following:

(a) $\lim\limits_{x\to -3^-} \dfrac{x^2|2x+6|}{4x+12}$ (b) $\lim\limits_{x\to 6^+} \dfrac{x^2-8x+12}{|2x-12|}$ (c) $\lim\limits_{x\to 7} \dfrac{(\sin x)|3x-21|}{x^2-49}$

Very often, we must evaluate limits involving trig functions, and very often, these limits are not easy to evaluate unless we have one of three things: (1) a calculator, (2) a smart, helpful friend, or (3) a couple of trig limits memorized.

Example 13:
With a calculator, evaluate the following, then memorize the results:

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} =$

Now try

$\lim\limits_{x\to 0} \dfrac{x}{\sin x} =$

$\lim\limits_{x\to 0} \dfrac{\sin 5x}{5x} =$

$\lim\limits_{x\to 0} \dfrac{\sin(x/2)}{x/2} =$

$\lim\limits_{x\to 4} \dfrac{x-4}{\sin(x-4)} =$

(b) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} =$

Now try

$\lim\limits_{x\to 0} \dfrac{x}{1-\cos x} =$

$\lim\limits_{x\to 0} \dfrac{1-\cos \pi x}{\pi x} =$

$\lim\limits_{x\to 0} \dfrac{1-\cos(x/9)}{x/9} =$

$\lim\limits_{x\to -7^-} \dfrac{1-\cos(x+7)}{x+7} =$

Useful Trig Limits

In general, if $g(c)=0$, then

$$\lim_{x\to c}\frac{\sin(g(x))}{g(x)}=1=\lim_{x\to c}\frac{g(x)}{\sin(g(x))} \quad \text{and} \quad \lim_{x\to c}\frac{1-\cos(g(x))}{g(x)}=0=\lim_{x\to c}\frac{\cos(g(x))-1}{g(x)}$$

This is due to the preservation of the behavior of the graphs at $x=c$ undergoing a horizontal dilation or horizontal shift

We can now use these memorized limits in conjunction with our limit properties. But WHAT if our expressions are slightly different??

Example 14:

(a) $\lim\limits_{x\to 0}\dfrac{\sin 7x}{6x}=$

(b) $\lim\limits_{x\to 0}\dfrac{3x}{\sin \pi x}=$

(c) $\lim\limits_{x\to 0}\dfrac{4-4\cos 11x}{44x}=$

Example 15:
Evaluate the following

(a) $\lim\limits_{x\to 0}\dfrac{x+\sin x}{x}$

(b) $\lim\limits_{x\to 0}\dfrac{x}{\tan x}$

(c) $\lim\limits_{\theta\to 0}\dfrac{\sec\theta-1}{\theta\sec\theta}$

(d) $\lim\limits_{x\to 0}\dfrac{4\sin 5x}{x\cos 2x}$

(e) $\lim\limits_{x\to 0}\dfrac{\tan 3x}{\sin 4x}$

(f) $\lim\limits_{x\to 0}\dfrac{\cot 11x}{\csc 25x}$

(g) $\lim\limits_{x\to 0}\dfrac{\cos 14x}{\csc 9x}$

(h) $\lim\limits_{x\to 0}\dfrac{\csc 7x}{\sin \pi x}$

(i) $\lim\limits_{t\to 0}\dfrac{\sin^2 2t}{3t^2}=$

MEMORIZE the following:

$$\lim_{x\to 0}\frac{\sin mx}{mx}=\lim_{x\to 0}\frac{mx}{\sin mx}=\lim_{x\to 0}\frac{mx}{\tan mx}=\lim_{x\to 0}\frac{\tan mx}{mx}=1$$

$$\lim_{x\to 0}\frac{\sin mx}{nx}=\lim_{x\to 0}\frac{mx}{\sin nx}=\lim_{x\to 0}\frac{mx}{\tan nx}=\lim_{x\to 0}\frac{\tan mx}{nx}=\frac{m}{n}$$

$$\lim_{x\to 0}\frac{\sin mx}{\sin nx}=\lim_{x\to 0}\frac{\tan mx}{\tan nx}=\lim_{x\to 0}\frac{\tan mx}{\sin nx}=\lim_{x\to 0}\frac{\sin mx}{\tan nx}=\frac{m}{n}$$

$$\lim_{x\to 0}\frac{\csc mx}{\csc nx}=\lim_{x\to 0}\frac{\cot mx}{\cot nx}=\lim_{x\to 0}\frac{\cot mx}{\csc nx}=\lim_{x\to 0}\frac{\csc mx}{\cot nx}=\frac{n}{m}$$

$$\lim_{x\to 0}\frac{1-\cos mx}{mx}=\lim_{x\to 0}\frac{\cos mx-1}{mx}=0$$

Example 16: <u>General Cleverness</u>
This is a great method when all else fails.

(a) $\lim_{x\to 0}\frac{5x^2}{1-\cos x}=$

(b) $\lim_{x\to 5}\frac{3x-15}{\sqrt{x^2-10x+25}}=$

(c) $\lim_{x\to \pi/4}\frac{1-\tan x}{\sin x-\cos x}=$

(d) $\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=$

(e) $\lim_{x\to \pi/3}\frac{12\cos^2 x+2\cos x-4}{4\cos x-2}=$