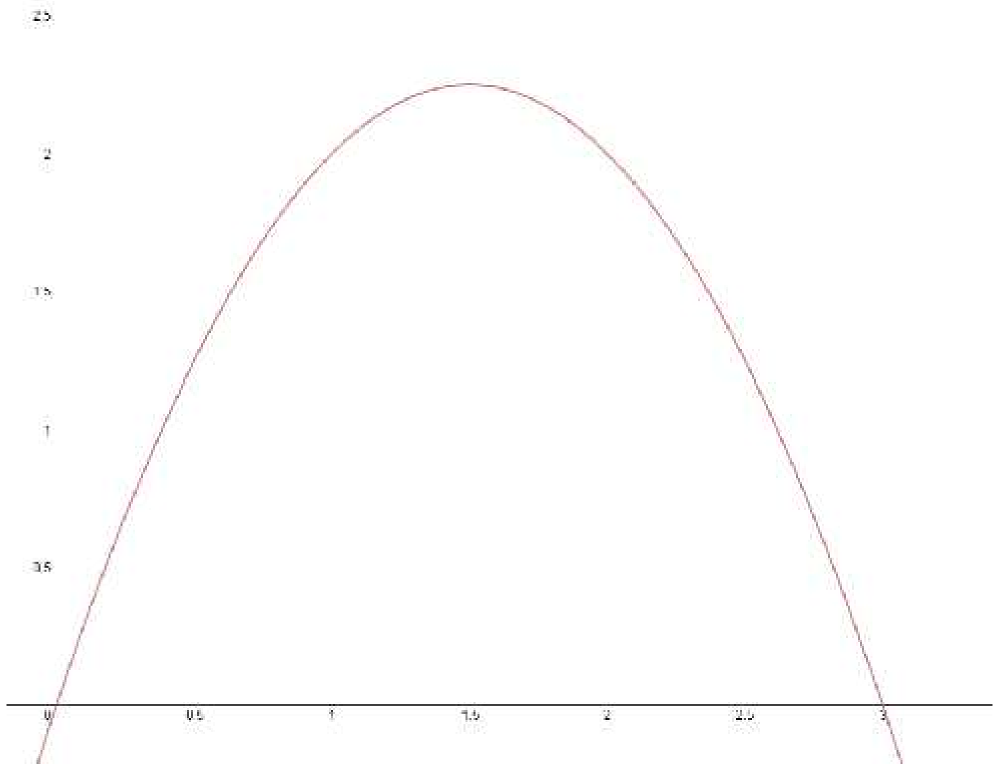
§2.1—Tangent Line Problem

Example 1:



The graph of the function $f(x)=3x-x^2$ is shown above. How fast is the graph of $f(x)$ changing at $x=1$, that is, what is the **slope of the tangent line** at $x=1$? Using your calculator, evaluate the average rates of change (**slopes of the secant lines**) between the point $(1, f(1))$ and the following points (be sure to show the difference quotient.

(a) $(1.5, f(1.5))$ (b) $(1.1, f(1.1))$ (c) $(1.01, f(1.01))$ (d) $(1.001, f(1.001))$

(e) Based upon your mathematical evidence above, how fast is the graph of $f(x)$ changing at $x=1$, that is, what is the **slope of the tangent line** at $x=1$?

Example 2:

For $f(x)=3x-x^2$,

(a) find the average rate of change between the points $(1, f(1))$ and $(1+h, f(1+h))$, where h is the change in x between our two x-values. Simplify your function, $A(h)$.

(b) What happens to the secant lines between the points $(1, f(1))$ and $(1+h, f(1+h))$, as h gets smaller and smaller?

(c) Evaluate $\lim_{h\to 0} A(h)=$

Example 3:

For $f(x)=3x-x^2$, using a similar method used above, find a general formula for finding the slope of the tangent line to the graph of f at any point $(x, f(x))$ using the points $(x, f(x))$ and $(x+h, f(x+h))$.

*Listen closely and you can hear Galileo grumbling in his grave!

The slope function found in the previous example called the **derivative function** of $f(x)$, or $f'(x)$ (read as **"*f* prime of *x*"**). It can be used to find the slope of the tangent line to a graph at a point. A function that has a derivative at a point is said to be **differentiable** at that point.

The Limit Definition of the Derivative of a Function

If $f(x)$ is a differentiable function, its derivative function, $f'(x)$, is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

NOTE: There are other notations for $f'(x)$. We will also use y' and $\frac{dy}{dx}$

These give you a variable function that can be used and reused for different *x*-values.

Example 4:

Given $f(x) = \sqrt{x-2}$

(a) Graph $f(x)$. Describe what the *y*-values are doing and ***how*** they are doing it.

(b) Find $f'(x)$ using the limit definition of the derivative.

(b) Find the slope of $f(x)$ at (i) $x = 6$ (ii) $x = 16$ (iii) $x = 27$ (iii) $x = 2$.

(c) Describe the values of *x* for which $f(x)$ is (i) continuous (ii) differentiable.

(d) Write the equation of the tangent line, in Taylor form, to $f(x)$ at $x = 27$.

Sometimes we might be interested in finding a slope of a function at a point directly.

Modified form of the limit definition of the derivative

The **numeric value** of the derivative of a function $f(x)$ at a point $(c, f(c))$ is given as

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Example 5:

For $f(x) = \dfrac{x}{x+1}$,

(a) Find $f'(1)$ directly using the modified definition of the derivative.

(b) Find the equation of the <u>normal</u> line to $f(x)$ at $x = 1$.

We will be referring to rates of change and slopes quite a bit from here on out. At this time, it's important to formally lay out the meanings and distinctions between the two different types of each.

Using 2 points, $(a, f(a))$ and $(b, f(b))$	**Using 1 point $(c, f(c))$**
The following are equivalent: • Slope of the secant line • $\dfrac{f(b) - f(a)}{b-a}$ • Average rate of change on the interval $[a,b]$	**The following are equivalent:** • Slope of the tangent line • $\lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ • $f'(c)$ • Derivative at $x = c$ • Instantaneous rate of change at $x = c$

Example 6:
Each of the following represents the derivative of some function $f(x)$ at some $x = c$. Identify the function $f(x)$ and c for each one.

(a) $\lim\limits_{h\to 0}\dfrac{\left[5(2+h)^2-(2+h)+7\right]-\left[5(2)^2-(2)+7\right]}{h}$

(b) $\lim\limits_{h\to 0}\dfrac{\tan\left(\frac{\text{®}}{4}+h\right)-1}{h}$

(c) $\lim\limits_{h\to 0}\dfrac{e^{\text{®}+h}-e^{\text{®}}}{h}$

(d) $\lim\limits_{h\to 0}\dfrac{\ln(1+h)}{h}$

Remember that if a function, $f(x)$, has a derivative at a given point $(x, f(x))$, then the function is **differentiable** at that point. We also saw in **Example 4** how the function $f(x)=\sqrt{x-2}$ was defined (and continuous) at $x = 2$, but was not differentiable at $x = 2$. **<u>It turns out, being continuous at a point is a necessary condition for differentiability at the same point, but not sufficient.</u>**

Intuitive definition of Differentiability at a Point

If a function, $f(x)$, is **<u>continuous</u>** as $x = c$, then it is also differentiable at $x = c$ if the slopes from either side of $x = c$ are approaching each other and are finite. That is, if

$$\lim_{x\to c^-} f'(x) = L = \lim_{x\to c^+} f'(x)$$

In order to check that BOTH the *y*-values AND the slopes are approaching each other from either side of $x = c$, we must define a new limit definition of $f'(x)$ as a function of *x*, rather than a function of *h*.

Example 7:
Let $f(x)$ be a continuous function. If $f'(x)$ is given be each of the following below, determine if $f(x)$ is differentiable at $x = -1$.

(a) $f'(x)=\begin{cases}3x+4, & x<-1\\ x^2, & x\ge -1\end{cases}$

(b) $f'(x)=\begin{cases}\ln(x+2), & x<-1\\ \cos(\text{®}x), & x\ge -1\end{cases}$

Alternate form definition of the derivative

The **numeric value** of the derivative of a function $f(x)$ at a point $(c, f(c))$ is ALSO given as

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

*NOTE 1: this version ALSO gives a numeric value, but now the limit is a function of *x*. This is important.
*NOTE 2: rather than letting the distance between our two poinst go to zero, this definition simply lets the second point move closer and closer to our desired point.

Example 8:

Verify Example 5, by finding $f'(1)$ for $f(x) = \frac{x}{x+1}$ using the alternate form.

Example 9:

Each of the following represents the derivative of some function $f(x)$ at some $x = c$. Identify the function $f(x)$ and *c* for each one.

(a) $\lim_{x \to 5} \frac{(x-2)^3 - 27}{x - 5}$

(b) $\lim_{x \to -3} \frac{\sqrt{x+9} - \sqrt{6}}{x + 3}$

Differentiability at a point $x = c$.

A function $f(x)$ is differentiable at $x = c$ if and only if $\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = L = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$, where L is a finite value.

*This basically says that to be differentiable at a point $(c, f(c))$, the graph must be continuous (connected) first, but must connect in a way that the slopes merge into each other smoothly.

SMOOOOOTHLY CONNECTED

Example 10:
Use the definition of differentiability at a point to determine whether the following function is differentiable at $x = 1$.

$$f(x) = \begin{cases} -3x + 1, & x \le 1 \\ x^2 - 2x - 1, & x > 1 \end{cases}$$

A function that is differentiable at a point is also said, synonymously, to be **Locally Linear** at that point. The idea is, that if you zoom in close enough on any differentiable curve at a specific point, that graph will begin to look more and more and more like its tangent line at that point. **IF THE SLOPE OF THAT LINE EXISTS**, the function will be differentiable at that point, and the slope will be its derivative value. You can think, then of a differentiable function to be made up of infinitely many, infinitely small, connected line segments (rather than dots). This will be an important idea when we study Slope Fields later on.

Example 11:
Using your calculator, zoom in to determine of the following function are locally linear at the indicated point.

(a) $f(x) = e^{5x}$ at $x = -\frac{1}{2}$

(b) $g(x) = |x - 2| + 1$ at $x = 2$

The consequence of local linearity and differentiability being logically equivalent means we have a visual way to determine whether the graph of a function is differentiable at a point or not.

Example 12:
Sketch a graph of each of the following, then determine if the function is differentiable at the indicated point.

(a) $f(x) = |2x+6|$ at $x = -3$

(b) $f(x) = \sqrt[3]{x}$ at $x = 0$

(c) $f(x) = x^{2/3}$ at $x = 0$

(d) $f(x) = \sqrt{10-x}$ at $x = 10$

Very Important Theorem:

If f is differentiable at $x = c$, then f is continuous at $x = c$.

$$D \to C$$

Contrapositive of Very Important Theorem:

If f is NOT continuous $x = c$, then f is NOT differentiable at $x = c$.

$$\neg C \to \neg D$$

It takes MORE than just the slopes to be approaching the same value on either side of a particular *x*-value for a function to be differentiable there. It must first be connected, THEN smooooothly so. One must ALWAYS be careful with piecewise functions.

Example 13:

For $f(x)=\begin{cases}2x-1, & x\le -1\\ 2x+1, & x>-1\end{cases}$, determine of $f(x)$ is differentiable at $x=-1$ by using

(a) the alternate form

(b) a continuity argument

There is one other form of the definition of the derivative worth mentioning.

Symmetric form definition of the derivative

The **numeric value** of the derivative of a function $f(x)$ at a point $(c, f(c))$ is ALSO given as

$$f'(c)=\lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h}$$

Your calculator can compute numeric derivatives for you. From your home screen, type in “Math” “8”

Example 14:

Using your calculator, if $f(x)=3-6|x+1|$, find the following

(a) $f'(-2)$

(b) $f'(0)$

(c) $f'(-1)$

*Your calculator’s answer on 13(c) is R-O-N-G rong! Your calculator is programmed using the symmetric difference quotient. It will lie to you sometimes. You are the calculator’s boss, not the other way around. YOU make the final call, Boss!