§2.10—Derivatives of Log Functions & LOG DIFF

Definition of a Log function

If \( b > 0 \) and \( b \neq 1 \), then \( f(x) = b^x \) is a one-to-one function, hence it is invertible, ergo it has an inverse called the \textit{logarithm base} \( b \). Additionally,

\[
\begin{align*}
  b^x &= y \\
  \Leftrightarrow \\
  \log_b y &= x
\end{align*}
\]

Recall that a log is simply an exponent, a special exponent. It is the exponent to which one must raise a particular base to achieve a desired result. A numeric log expression such as \( \log_4 16 \) can be interpreted as \textit{“Four raised to what power is sixteen.”} The answer, of course, is two. So

\[ \log_4 16 = 2 \text{ because } 4^2 = 16 \]

Example 1:
Evaluate the following quickly:

\begin{align*}
  (a) \quad \log_2 8 & \quad (b) \quad \log_3 \frac{1}{81} & \quad (c) \quad \log_{16} 4
\end{align*}
Example 2:
Sketch the graph of a log function \( f(x) = \log_b x \) for the following. State the domain and range.

(a) \( b > 1 \)  
(b) \( 0 < b < 1 \)

Recall:
- \( \ln x = \log_e x \)
- \( \log x = \log_{10} x \)
- \( x \) is called the argument

Example 3:
Use the properties of limits to find the limit of each of the following:

(a) \( \lim_{x \to \infty} \ln(x^2 - x) \)  
(b) \( \lim_{x \to 9^+} \log_2 (x - 9) \)  
(c) \( \lim_{x \to \pi^+} \log(\tan x) \)  
(d) \( \lim_{x \to 0^+} \log(\cos x) \)

Before we delve into the calculus of logs, we better be sure we can find domains of log functions. It sure would be silly finding a derivative where the graph doesn’t even exist. Don’t be that guy!

Example 4:
For \( g(x) = \log(x^2 - 3x + 2) \) find

(a) The domain of \( g(x) \)  
(b)(i) \( \lim_{x \to 1^+} g(x) \)  
(ii) \( \lim_{x \to 1^-} g(x) \)  
(iii) \( \lim_{x \to 2^+} g(x) \)

(c) (i) \( \lim_{x \to \infty} g(x) \)

(d) What would the answers to all the above be if the function were \( g(x) = \log_{0.1}(x^2 - 3x + 2) \)?
What makes logs so useful is also what makes them so much fun.

**Properties of Logarithms**

- \( \log_b (MN) = \log_b M + \log_b N \)
- \( \log_b \frac{M}{N} = \log_b M - \log_b N \)
- \( \log_b M^r = r \log_b M \)
- \( \log_b b^x = x = b^{\log_b x} \)
- If \( \log_b M = \log_b N \), then \( M = N \)

**Example 5:**
Use the properties of logs to solve the following equations for \( x \).

(a) \( \log(x+1) = 3 \)  
(b) \( e^{3-x} = 14 \)

(c) \( \ln x - \ln(x-1) = 1 \)  
(d) \( \log x + \log(x+1) = \log 6 \)

Time to B.O.T.C. (Bust Out The Calculus)

**Example 6:**
If \( y = \ln x \), derive \( \frac{dy}{dx} \), the derivative of \( y = \ln x \). Verify your answer by sketching both \( y = \ln x \) and \( \frac{dy}{dx} \) on the same coordinate plane.
Example 7:
If \( f(x) = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases} \), sketch \( f(x) \) on the same coordinate plane, then attempt to sketch the graph of \( f'(x) \). Verify your graph by using your result from Example 6 and the chain rule to find \( f'(x) \).

MEMORIZE

\[
\frac{d}{dx}[\ln x] = \frac{1}{x} \quad \frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \frac{d}{dx}[\ln u] = \frac{1}{u} u' \text{ (chain rule)}
\]

We can now combine our new log rules with all the other rules we have learned.

Example 8:
Find the derivative of each of the following. Remember to simplify early and often, especially when you have logs!

(a) \( h(x) = \cos(\ln x) \)

(b) \( y = \ln(1 + \ln x) \)

(c) \( f(u) = \ln \sqrt{\frac{3u + 2}{3u - 2}} \)

Sooo, uhhhh . . . what if the base of the log is not \( e \)? Nothin’ a li’l Change-o’-Base formula won’t remedy.

Example 9:
If \( y = \log_b x \), use the change of base formula to find \( \frac{dy}{dx} \).


**MEMORIZE**

\[
\frac{d}{dx} \left[ \log_b x \right] = \frac{1}{x} \cdot \frac{1}{\ln b} = \frac{1}{x \ln b} \quad \quad \frac{d}{dx} \left[ \log_b |x| \right] = \frac{1}{x \ln b} \quad \quad \frac{d}{dx} \left[ \log_b u \right] = \frac{1}{u \ln b} \cdot u' \quad \text{(chain rule)}
\]

**Example 10:**

If \( f(x) = \left( \log_3 (5 - x^4) \right)^2 \), find \( f'(x) \).

---

We are rounding third base and heading home when it comes to logs and derivatives. There’s one last application of logs we learn before we slide safely into home.

**Logarithmic Differentiation (or LOG DIFF once you master it)**

This technique involves taking the natural log of BOTH sides of an equation prior to differentiating. We can use this technique in three situations:

1. When differentiating an “ugly” expression with lots of factors in a product and/or a quotient, thereby making the derivative easier to compute (Simplify early and often, ESPECIALLY WHEN YOU HAVE LOGS!)
2. When differentiating a function of the form \( y = f(x)^{g(x)} \). (Hybrid power/exponential function).
3. When the instructions say “Use Logarithmic Differentiation to . . . “

**Example 11:**

Find the derivative of each of the following

(a) \( y = \frac{e^x \left( x^2 + 2 \right)^4}{\sqrt{(x+1)^3 \left( x^2 + 3 \right)^2}} \)

(b) \( \frac{d}{dx} \left[ (\sin x)^{\cos x} \right] = \)}
Example 12:
If \( y = x^2 \), find \( \frac{dy}{dx} \) using logarithmic differentiation.

Bonus: Other Uses For Logs.