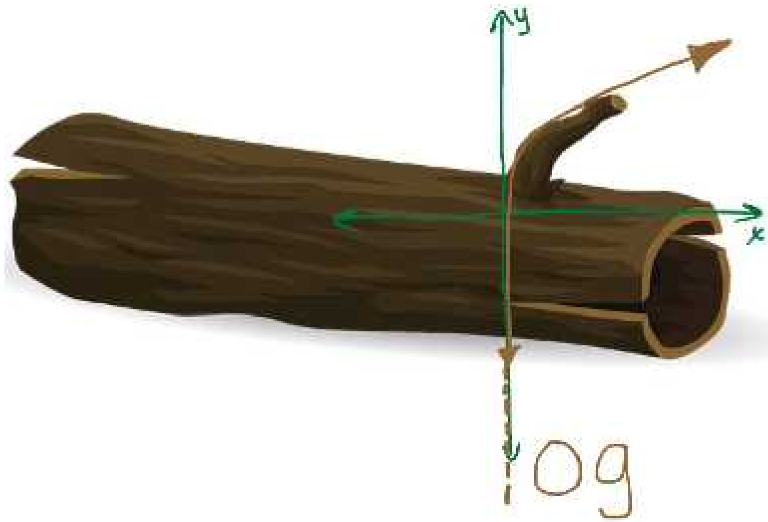
§2.10—Derivatives of Log Functions & LOG DIFF



Definition of a Log function

If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is a one-to-one function, hence it is invertible, ergo it has an inverse called the **logarithm base *b***. Additionally,

$$b^x = y$$
$$\Leftrightarrow$$
$$\log_b y = x$$

Recall that a log is simply an exponent, a special exponent. It is the exponent to which one must raise a particular base to achieve a desired result. A numeric log expression such as $\log_4 16$ can be interpreted as "***Four raised to what power is sixteen.***" The answer, of course, is two. So

$$\log_4 16 = 2 \text{ because } 4^2 = 16$$

Example 1:
Evaluate the following quickly:

(a) $\log_2 8$ (b) $\log_3 \frac{1}{81}$ (c) $\log_{16} 4$

Example 2:

Sketch the graph of a log function $f\left(x\right)=\log_b x$ for the following. State the domain and range.

(a) $b>1$

(b) $0<b<1$

Some logs become dugout canoes. Some dugout canoes don't get finished and become firewood.

Recall:

- $\ln x=\log_e x$
- $\log x=\log_{10} x$
- x is called the **argument**

Example 3:

Use the properties of limits to find the limit of each of the following:

(a) $\lim\limits_{x\to\infty}\ln(x^2-x)$

(b) $\lim\limits_{x\to 9^+}\log_2\left(x-9\right)$

(c) $\lim\limits_{x\to\frac{\text{®}}{2}^-}\log\left(\tan x\right)$

(d) $\lim\limits_{x\to 0^+}\log\left(\cos x\right)$

Before we delve into the calculus of logs, we better be sure we can find domains of log functions. It sure would be silly finding a derivative where the graph doesn't even exist. Don't be that guy!

Example 4:

For $g\left(x\right)=\log\left(x^2-3x+2\right)$ find

(a) The domain of $g\left(x\right)$

(b)(i) $\lim\limits_{x\to 1^+}g\left(x\right)$

(ii) $\lim\limits_{x\to 1^-}g\left(x\right)$

(iii) $\lim\limits_{x\to 2^+}g\left(x\right)$

(c) (i) $\lim\limits_{x\to\infty}g\left(x\right)$

(ii) $\lim\limits_{x\to-\infty}g\left(x\right)$

(d) What would the answers to all the above be if the function were $g\left(x\right)=\log_{0.1}\left(x^2-3x+2\right)$?

What makes logs so useful is also what makes them so much fun.

Properties of Logarithms

• $\log_b(MN) = \log_b M + \log_b N$ • $\log_b \frac{M}{N} = \log_b M - \log_b N$ • $\log_b M^r = r\log_b M$	• $\log_b b^x = x = b^{\log_b x}$ • If $\log_b M = \log_b N$, then $M = N$

Example 5:
Use the properties of logs to solve the following equations for *x*.

(a) $\log(x+1) = 3$

(b) $e^{3-x} = 14$

(c) $\ln x - \ln(x-1) = 1$

(d) $\log x + \log(x+1) = \log 6$

Time to B.O.T.C. (Bust Out The Calculus)

Example 6:
If $y = \ln x$, derive $\frac{dy}{dx}$, the derivative of $y = \ln x$. Verify your answer by sketching both $y = \ln x$ and $\frac{dy}{dx}$ on the same coordinate plane.

Example 7:

If $f(x) = \ln|x| = \begin{cases} \ln x, & x > 0 \\ \ln(-x), & x < 0 \end{cases}$, sketch $f(x)$ on the same coordinate plane, then attempt to sketch the graph of $f'(x)$. Verify your graph by using your result from Example 6 and the chain rule to find $f'(x)$.

MEMORIZE

$\frac{d}{dx}[\ln x] = \frac{1}{x}$	$\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}$	$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot u'$ (chain rule)

We can now combine our new log rules with all the other rules we have learned.

Example 8:

Find the derivative of each of the following. **Remember to simplify early and often, ESPECIALLY WHEN YOU HAVE LOGS!**

(a) $h(x) = \cos(\ln x)$ (b) $y = \ln(1 + \ln x)$ (c) $f(u) = \ln\sqrt{\frac{3u+2}{3u-2}}$

Sooo, uhhhh . . . what if the base of the log is not e ? Nothin' a li'l Change-o'-Base formula won't remedy.

Example 9:

If $y = \log_b x$, use the change of base formula to find $\frac{dy}{dx}$

MEMORIZE

$\frac{d}{dx}\left[\log_b x\right] = \frac{1}{x}\cdot\frac{1}{\ln b} = \frac{1}{x\ln b}$ $\frac{d}{dx}\left[\log_b |x|\right] = \frac{1}{x\ln b}$ $\frac{d}{dx}\left[\log_b u\right] = \frac{1}{u\ln b}\cdot u'$ (chain rule)

Example 10:

If $f(x) = \left(\log_3\left(5-x^4\right)\right)^2$, find $f'(x)$.

We are rounding third base and heading home when it comes to logs and derivatives. There's one last application of logs we learn before we slide safely into home.

Logarithmic Differentiation (or LOG DIFF once you master it)

This technique involves taking the natural log of BOTH sides of an equation prior to differentiating. We can use this technique in three situations:

1. When differentiating an **"<u>ugly</u>"** expression with lots of factors in a product and/or a quotient, thereby making the derivative easier to compute (**Simplify early and often, ESPECIALLY WHEN YOU HAVE LOGS!**)
2. When differentiating a function of the form $y = f(x)^{g(x)}$. (Hybrid power/exponential function).
3. When the instructions say "Use Logarithmic Differentiation to . . . "

Example 11:

Find the derivative of each of the following

(a) $y = \dfrac{e^x\left(x^2+2\right)^4}{\sqrt{(x+1)^3}\left(x^2+3\right)^2}$

(b) $\dfrac{d}{dx}\left[(\sin x)^{\cos x}\right] =$

Example 12:

If $y = x^2$, find $\frac{dy}{dx}$ using logarithmic differentiation.

Bonus: Other Uses For Logs.

Coffee Tables

Planters

Placemats

Trees