§2.3—Differentiation Rules

- $\frac{dy}{dx}$ is a **noun**. It means "the derivative of *y* with respect to *x*."
- $\frac{d}{dx}$ is a **verb**. It means "take the derivative with respect to *x*" of the expression that follows.

The Constant Rule

The derivative of a constant function is 0. That is, if *c* is a real number, then $\frac{d}{dx}[c]=0$. The derivative of $y=c$ is $\frac{dy}{dx}=0$.

Example 1:
Find the derivative of the following functions:

(a) $y=8$ (b) $f(x)=0$ (c) $s(t)=3$ (d) $y=k\pi^2$, *k* is a constant

The Power(ful) Rule

If *n* is a real number and *a* is some constant in the function $f(x)=ax^n$, then $\frac{d}{dx}\left[ax^n\right]=anx^{n-1}$. Equivalently, $f'(x)=anx^{n-1}$.

Example 2:
Find the derivative of the following functions:

(a) $f(x)=2x^3$ (b) $g(x)=\frac{\sqrt[3]{x}}{3}$ (c) $y=\frac{5}{3x^\pi}$ (d) $y=\frac{6}{\sqrt[5]{x^3}}$

Example 3:

If $f(x)=\frac{x^4}{2}$, find each of the following.

(a) $f'(-1)$ (b) $f'(0)$ (c) The x-coordinate where f has a slope of 128.

Example 4:

If $f(x)=3x^2$

(a) find the equation of the tangent line at $x=-2$.

(b) find the equation of the normal line at $x=-2$.

(c) find the points where the normal line intersects the graph of $f(x)=3x^2$.

Rewriting is very important when using the Power Rule. This is worth repeating. Rewriting is very important when using the Power Rule. An expression MUST be in the form ax^n and n MUST be a real number.

Example 6:

Rewrite, evaluate (differentiate), and then simplify the following:

(a) $\frac{d}{dx}\left[\frac{5}{2x^{\sqrt{2}}}\right]$ (b) $\frac{d}{dx}\left[\frac{4}{(2x)^3}\right]$ (c) $\frac{d}{dt}\left[\frac{7t}{3\sqrt{t}}\right]$ (d) $\frac{d}{dm}\left[\frac{6}{(3m)^{-2}}\right]$

The Sum and Difference & Konstant Rules

The derivative of the sum of two functions *f* and *g* is the sum of the derivatives of *f* and *g*. Similarly, the derivative of the difference of two functions *f* and *g* is the difference of the derivatives of *f* and *g*.

$$\frac{d}{dx}\left[f(x)\pm g(x)\right]=f'(x)\pm g'(x)$$ Addition/Subtraction Rule

$$\frac{d}{dx}\left[kf(x)\right]=k\frac{d}{dx}\left[f(x)\right]$$ Konstant Rule

Example 7:
Find the derivative of the following functions:

(a) $f(x)=x^3-4x+5$

(b) $g(x)=-\dfrac{x^4}{2}+2x^3-5x$

(c) $y=\dfrac{2x^3-3x^2+7x+5}{2\sqrt{x}}$

(d) $y=x(3x+2)^2$

A powerful application of the derivative is finding the values of *x* where a function has horizontal tangent lines.

Example 8:
Find the **coordinates** (x,y) and **equations** of any horizontal tangents to the curve $y=x^4-2x^2+2$.

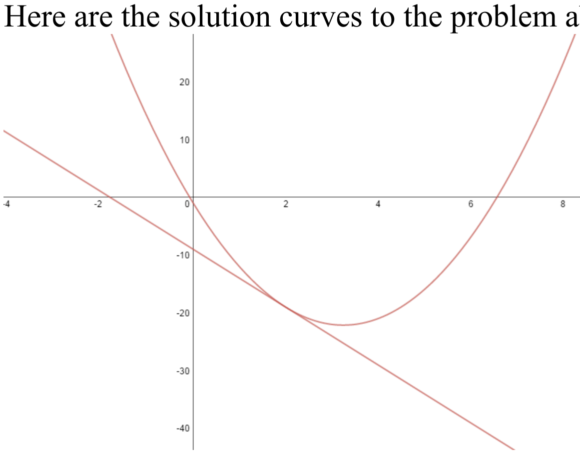
It's important to remember that at any point of tangency, $x = c$, a function $f(x)$ and its tangent line $T(x)$ share two things:

1. A *y*-value
2. A slope value

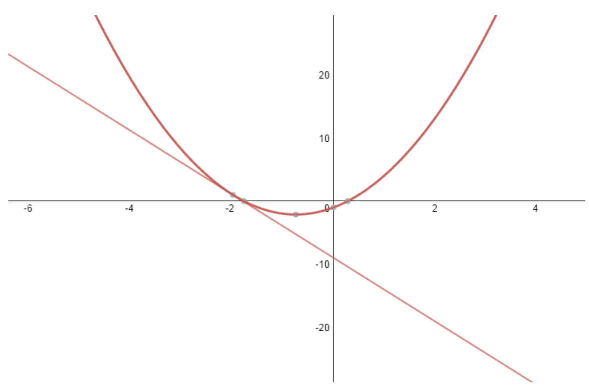
Here's a nice application involving this important idea.

Example 9:
Find the value(s) of *k* such that the line $5x + y = -9$ is tangent to the graph of $f(x) = 2x^2 + kx - 1$.

Here are the solution curves to the problem above.



$f(x) = 2x^2 - 13x - 1$



$f(x) = 2x^2 + 3x - 1$

In the previous section, you recognized graphically the derivative of sine. It's time to memorize two, of what will be many, many, many derivative rules.

The Sine and Cosine Rules

$$\frac{d}{dx}[\sin x]=\cos x \qquad \text{and} \qquad \frac{d}{dx}[\cos x]=-\sin x$$

Example 10:
Find the derivatives of the following functions:

(a) $y=2\sin(x)+7$

(b) $f(x)=x+\cos x$

(c) $g(x)=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

Example 11:
If $f(x)=\sin x$, find the following.

(a) The slope of the graph of $f(x)$ at $x=0$.

(b) $f'\left(\frac{7\pi}{6}\right)$

(c) All $x\in[0,2\pi)$ where the graph of *f* has horizontal tangent lines.

Here's an application of the power rule of which Newton would be proud—Motion!

We know now that the derivative of a function at a point gives us two equivalent things:

1. The slope of the tangent line at that point
2. The instantaneous rate of change at that point

If we let $h(t)$ be a position function as a function of time *t*, $v(t)$ be a velocity function as a function of time, and $a(t)$ be an acceleration function as a function of time, we can relate these instantaneous function to each other in the following way.

$$h''(t) = v'(t) = a(t)$$

Example 12:
A calculus textbook is stolen from a museum and taken to the top of a 400-foot tower to be dropped down to the unforgiving pavement below. Its height $h(t)$ feet at any time *t* seconds during its brief, but painful, plummet to Earth, is given by the equation $h(t) = -16t^2 + 400$. Answer the following questions regarding the book.

(a) For how many actual seconds was the book falling? How many seconds did this feel like to the book?

(b) What was the velocity of the book, in ft/sec, the instant it hit the pavement? At this velocity, what happened to the page in the book discussing the Squeeze Theorem?

(c) What was the average velocity of the book, in ft/sec for the duration of its free-fall?

(c) What was the acceleration of the book, in ft/sec/sec during its fall?

Now that we know the power rule, we can circumvent the alternate form of the derivative to answer questions regarding differentiability such as this one (and the next one).

Example 13:

If $f(x)=\begin{cases}3-x, & x<1\\ ax^2+bx, & x\geq 1\end{cases}$, find the values of *a* and *b* such that $f(x)$ differentiable for all *x*.

Example 14:

If $f(x)=\begin{cases}bx^2-3, & x\leq -1\\ ax+b, & x>-1\end{cases}$, find the values of *a* and *b* such that $f(x)$ differentiable for all *x*.