

## §2.7—Implicit Differentiation

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- $\frac{d}{dx}[y] = \frac{dy}{dx}$
- $\frac{d}{dx}[x] = \frac{dx}{dx} = 1$

Sometimes we may be interested in finding the derivative of an equation that is not solved or able to be solved for a particular dependent variable explicitly. In this case, we must develop a way to analyze that variable's rate of change implicitly.

The equations below are equivalent, but are in different forms with respect to the dependent variable  $y$ .

**Explicit form**

$$y = \frac{2}{x}$$

**Implicit form**

$$xy = 2$$

**Example 1:**

Find the derivative,  $\frac{dy}{dx}$ , of  $y = \frac{2}{x}$ , an equation explicitly solved for  $y$  by taking the derivative of both sides with respect to  $x$ .

**Example 2:**

Find the derivative,  $\frac{dy}{dx}$ , of  $xy = 2$ , an equation NOT explicitly solved for  $y$  by taking the derivative of both sides with respect to  $x$ . Solve for  $\frac{dy}{dx}$ .

The method used to find  $\frac{dy}{dx}$  in Example 2 is called **Implicit Differentiation**. What do you notice about the implicit result compared to the result from Example 1? Which method was easier/faster?

In the first two examples, we had the option of differentiating explicitly or implicitly, but most of the time, we will use implicit differentiation when we're dealing with equations of curves that are not functions of a single variable, whose equations have powers of  $y$  greater than 1 making it difficult or impossible to explicitly solve for  $y$ . For such equations, we will be forced to use implicit differentiation, then solve for  $\frac{dy}{dx}$ , **which will be a function of either  $y$  alone or both  $x$  and  $y$ .**

**Important note 1:** Just because an equation is not explicitly solved for a dependent variable doesn't mean it can't. Your first step is to analyze whether it can be solved explicitly. This will make the problem a bit easier and your derivative will be a function of a single variable.

**Important note 1.5:** When differentiating implicitly, you must show that you are taking the derivative of both sides with respect to  $x$ .

$$\frac{d}{dx}[\text{Left Side}] = \frac{d}{dx}[\text{Right Side}]$$

**Important note 2:** It matters not if you use  $\frac{dy}{dx}$  or  $y'$ , just be careful not to mistake  $y'$  for  $y^1 = y$ .  $\frac{dy}{dx}$  stands out more prominently, but  $y'$  is a wee tad little bit faster.

### Example 3:

For the equation  $y = x + \sec(y)$ , find  $\frac{dy}{dx}$ .

In Example 3, we were asked to find the variable equation for  $\frac{dy}{dx}$ . Sometimes, however, we are interested only in finding the slope of a curve at given point. For such a problem, we can save a bit of algebraic manipulation if we plug in (and indicate so) our point earlier.

### Example 4:

Find the slope of the graph of  $y^3 + y^2 - 5y - x^2 = -4$  at  $(1, -3)$  by finding  $\frac{dy}{dx}$  first, then evaluating  $\frac{dy}{dx}$  at  $(1, -3)$

**Example 5:**

Find the slope of the graph of  $y^3 + y^2 - 5y - x^2 = -4$  at  $(1, -3)$  by differentiating then plugging in  $(1, -3)$  before solving for  $\frac{dy}{dx}$ .

Notice that in Examples 4 and 5, because our derivative equation had both an  $x$  and a  $y$  in it, we needed the actual ordered pair  $(x, y)$  to evaluate  $\frac{dy}{dx}$ . Often when only an  $x$ -value is given, and not an ordered pair, it is sign that our given equation can be explicitly solved for a single equation of  $y$ , like in Examples 1 and 2, but not always.

**Example 6:**

Find  $\frac{dy}{dx}$  at  $x=1$  for the equation  $y^2 + x = 2xy$

When your derivative  $\frac{dy}{dx}$  is a quotient of two variable expressions, then

- Horizontal tangents occur when  $\frac{dy}{dx} = \frac{0}{\neq 0}$
- Vertical tangents occur when  $\frac{dy}{dx} = \frac{\neq 0}{0}$
- No tangent line exists when  $\frac{dy}{dx} = \frac{0}{0}$  (these points must be thrown out)

**Example 7:**

The graph of the equation  $x^2 + y^2 = 4$  is a circle centered at the origin with a radius of four.

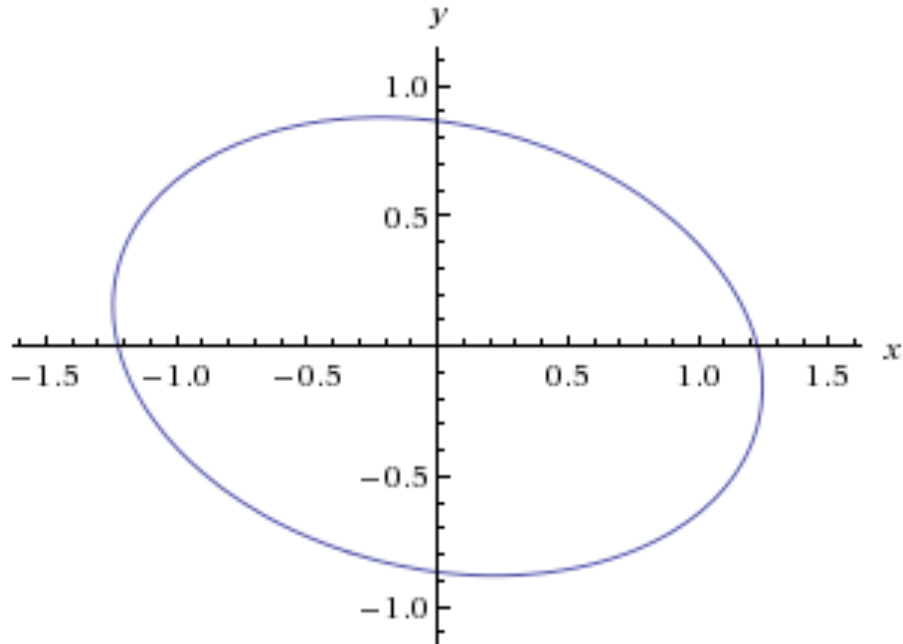
(a) Sketch the graph of the equation

(a) find  $\frac{dy}{dx}$ .

(b) Using calculus, verify that the graph has horizontal tangents at the point  $(0, 2)$  and  $(0, -2)$ .

(c) Using calculus, verify that the graph has vertical tangents at the point  $(2, 0)$  and  $(-2, 0)$ .

(d) At what value(s) of  $x$  is the slope of the graph  $\frac{3}{4}$ ?

**Example 8:**

*Graph generated by WolframAlpha*

The equation for the graph of the rotated ellipse shown above is  $2x^2 + xy + 4y^2 = 3$ ,

(a) Determine the  $x$ -value(s) of any horizontal tangent lines to the graph of  $2x^2 + xy + 4y^2 = 3$ .

(b) Determine the  $y$ -value(s) of any vertical tangent lines to the graph of  $2x^2 + xy + 4y^2 = 3$ .

**Example 9:**

Determine the equation of the tangent line of  $3(x^2 + y^2)^2 = 100xy$  at the point  $(3,1)$

When asked to find a higher-order derivative where implicit differentiation is needed, it is always beneficial to solve for  $\frac{dy}{dx}$  prior to finding the second derivative and beyond. This will always be possible because the first derivative will be a linear function of  $\frac{dy}{dx}$ . Subsequent derivatives will have a  $\frac{dy}{dx}$  visible in the equation, at which time, a substitution may be made to put your final derivative in terms of  $x$  and  $y$ .

**Example 10:**

Find  $\frac{d^2y}{dx^2}$  as a function of  $x$  and  $y$  if  $2x^3 - 3y^2 = 8$ .