§2.8—Inverse & Inverse Trig Functions

Have you ever done something you wish you hadn't? What if you could "undo" it? With some functions, you can!

Minnie and Mickey are one-to-one

Recall that a function possesses a unique inverse function if it, itself, is **one-to-one**.

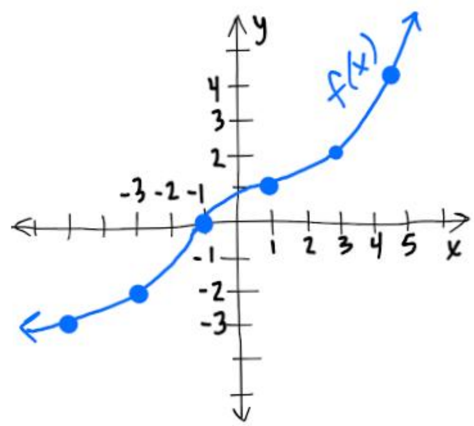
That is, for each *x*, there is only one *y* (if any) AND for each *y*, there is but one true love, *x*.

The graph of such functions pass the **horizontal line test**, and the function is said to be **invertible**.

The inverse function of $f(x)$ is denoted as $f^{-1}(x)$, and it "**undoes**" the work of a function $f(x)$.

Example 1:
Find the inverse functions of each of the following functions (if possible):

(a) $f:\{(2,5),\ (3,6),\ (-1,2),\ (4,7)\}$ (b) $f(x)=5x-8$ (c)



What do you notice about the slopes of the lines *f* and f^{-1} in part (b)?? Hmmmmmmmmmmm…

Notice also how the *x* and *y* information in each of the examples above interchanged between the function and its inverse. This is true for all functions and their inverses. Often, we name an inverse function $f^{-1}(x)$ as $g(x)$ for simplicity's sake. If we then let $f^{-1}(x)=g(x)$, then **the domain of f is the range of *g*, and the domain of *g* is the range of *f***. *x*-values of *f* become *y* values of *g* and verse-viça. A vertical asymptote of *f* becomes a horizontal asymptote of *g* and likewise in reverse.

$f(a)=b$ means $g(b)=a$. We call the ordered pairs $f:(a,b)$ and $g:(b,a)$ **corresponding points.**

Very often, you need to be able to recognize two functions as inverses without explicitly being told that they are. Here's a very important property of inverses that should help.

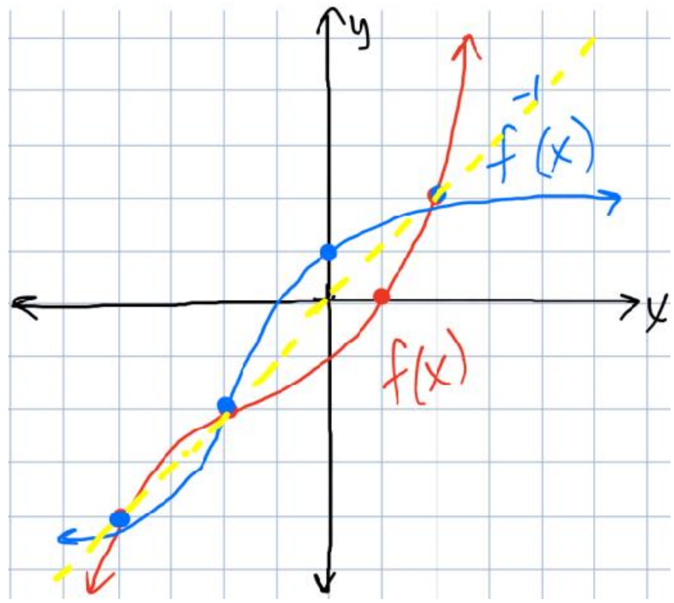
A function $g(x)$ is said to be the **inverse function** of the function $f(x)$ if $f(g(x)) = x = g(f(x))$ for all x in the domain of f and g.

Makes perfect sense: If you start with a quantity x and apply a function and its inverse back to back in either order, they "undo" themselves, and you're back where you started, x.

So where does the calculus come in? Remember Example 1 part (b)? Let's tip-toe closer to the answer.

Graphical consequence of Inverse Functions:

The graphs of a function $f(x)$ and its inverse $f^{-1}(x)$ are reflections of each other across the line $y = x$.



So what about slopes of inverse functions?

- **Inverse functions, at corresponding points, have reciprocal slopes.**
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We can state this mathematically like this:

Let f and g be inverse functions, such that $f(g(x)) = x = g(f(x))$ with $f(a) = b$ and $g(b) = a$, then

$$g'(b) = \frac{1}{f'(a)}$$

For instance, on the graph above, we know that $f:(1,0)$ and $f^{-1}:(0,1)$. If the slope of f at $x = 1$ is, say, 2, then the slope of f^{-1} at $x = 0$ is $\frac{1}{2}$. Written in calculus:

$$\left(f^{-1}\right)'(0) = \frac{1}{f'(1)}$$

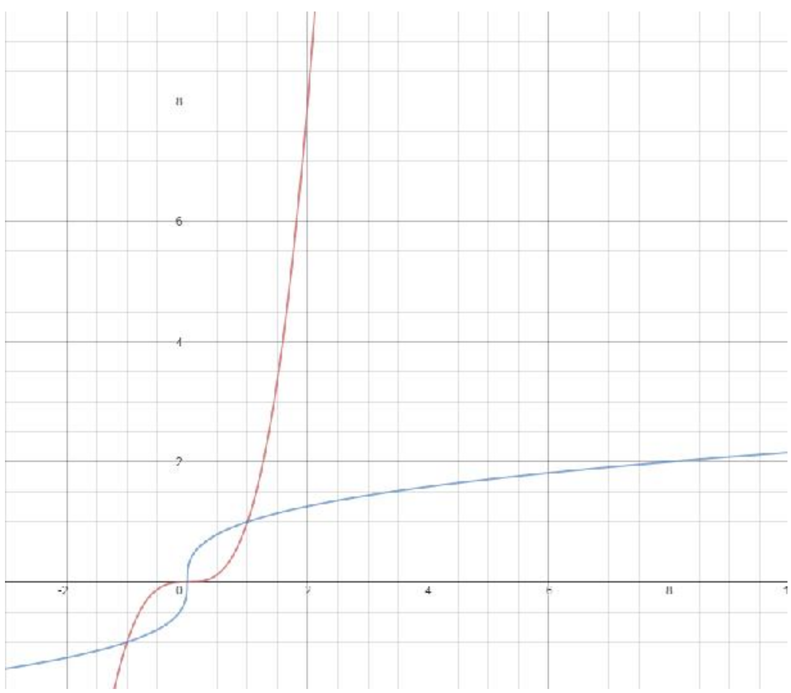
Before trying this out, it's worth mentioning that **inverse functions, at corresponding points, have reciprocal slopes.**

Example 2:

If $f(x)=x^3$, a one-to-one function, find the following:

(a) $f(2)$ (b) $f'(x)$ (c) $f'(2)$ (d) $f^{-1}(x)$

(e) $\left(f^{-1}\right)'(x)$ (f) $\left(f^{-1}\right)'(8)$ (g) $\left(f^{-1}\right)'(8)$ (h) $\dfrac{1}{f'(2)}$



In the previous example, we were able to find the inverse function explicitly, after which, we could take the derivative and plug in, but this won't always be the case. Part (g) above illustrates that we don't NEED to find the inverse function to evaluate its rate of change at a point.

Example 3:

If $f(x) = x^5 + 2x - 1$

(a) use $f'(x)$ and the continuity of $f(x)$ to show that $f(x)$ is one-to one

(b) find $f(1)$

(c) find $\left(f^{-1}\right)'(2)$.

Example 4:

If $f(x) = \frac{2}{5}x^5 + \frac{1}{3}x^3 + 3x + 2$, find (a) $f(0)$ and (b) $\left(f^{-1}\right)'(2)$

Sometimes we are handed all the ingredients to the recipe given to us in pre-measured bowls. All we have to do is dump them in and stir.

Example 5:

If $m(n(x)) = x = n(m(x))$, and if $m(5) = -9$, $m'(5) = -\frac{8}{11}$, then find $n'(-9)$.

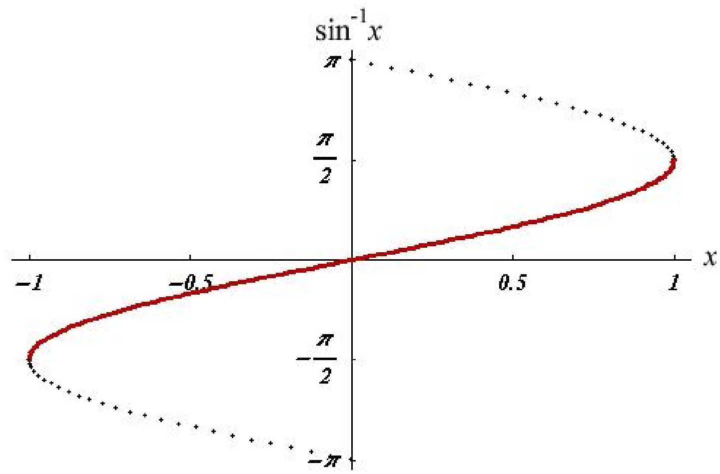
Example 6:

x	$f(x)$	$f'(x)$	$g(x)$
-3	4	6	7
4	-8	-2	-3
7	-3	2	9

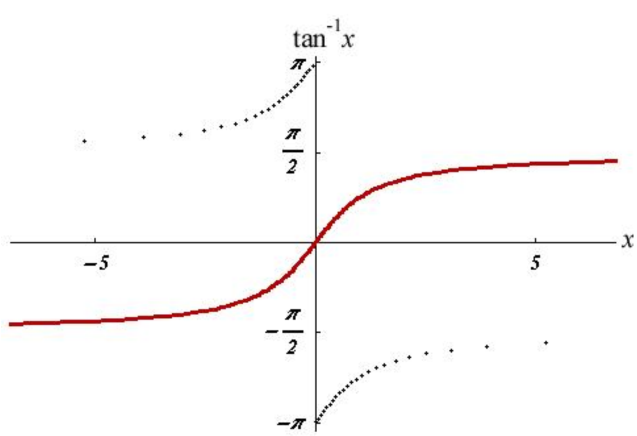
Selected values of one-to-one differentiable functions *f* and *g* (and f') are given in the table above. If $f(g(x)) = x = g(f(x))$, find $g'(-3)$.

None of the six trig functions is one-to-one, so to create inverse functions, their domains must be restricted to a convenient one-to-one interval covering all the ratio values. The three main inverse trig functions are below.

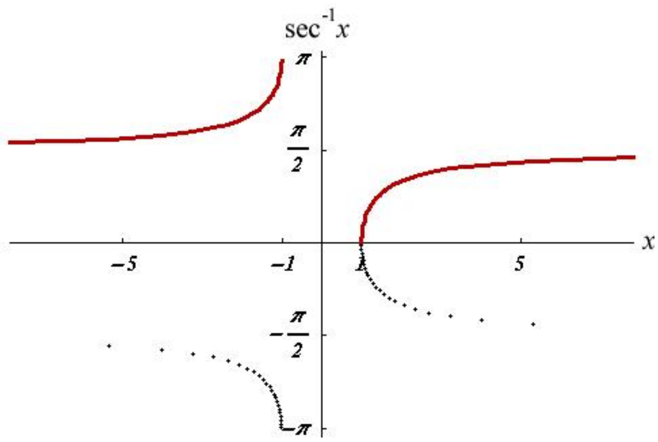
$$f(x) = \sin^{-1} x \text{ or } f(x) = \arcsin x$$



$$f(x) = \tan^{-1} x \text{ or } f(x) = \arctan x$$



$f(x) = \sec^{-1} x$ or $f(x) = \operatorname{arcsec} x$



I am really curious what the slopes of the tangent lines of these inverse trig functions are at various points in the domain. Are you?

We will now slake our curiosity by deriving the derivative of one of these by looking at it from a different perspective and using something we already know how to do: implicit differentiation.

Example 7:

If $y = \arcsin x$, find $\frac{dy}{dx}$

The Derivatives of the six Inverse Trig Functions (MEMORIZE (if you haven't already))

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2} \qquad \frac{d}{dx}[\operatorname{arccot} x] = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} x] = \frac{-1}{|x|\sqrt{x^2-1}}$$

Example 8:

Find the equation of the tangent line to the graph of $y = \cot^{-1} x$ at $x = 1$

These new derivative rules can be combined with the ones we already know to create new and exciting differentiation opportunities.

Example 9:

Differentiate:

(a) $\frac{d}{dx}\left[4x + \arcsin\left(2x^2\right)\right]$ (b) $\frac{d}{dx}\left[x \arctan\sqrt{x}\right]$ (c) $\frac{d}{dx}\left[\frac{\operatorname{arcsec}\left(\sin 3x\right)}{5}\right]$

Example 10:

Differentiate $y = \arcsin x + x\sqrt{1 - x^2}$, then simplify to a single term.

Example 11:
Find the coordinate of any horizontal **tangent** and find the equation of any horizontal **asymptotes** of $g(x) = (\arctan x)^2$

Sometimes it is useful to use an identity replacement when dealing with compositions of trig and inverse trig functions.

Example 12:
Rewrite $\tan\left(\arccos\left(x^2\right)\right)$ as an algebraic expression

For multiple choice questions, a quick preview of the answer choices can guide your approach to working the problem.

Example 13:
(Multiple Choice) Find the derivative of $f(x) = \cos\left(\tan^{-1}\frac{x}{\sqrt{6}}\right)$.

(A) $f'(x) = \frac{\sqrt{6}x}{\left(x^2+6\right)^{3/2}}$

(B) $f'(x) = \frac{x}{\left(x^2+6\right)^{3/2}}$

(C) $f'(x) = -\frac{\sqrt{6}}{\left(x^2+6\right)^{3/2}}$

(D) $f'(x) = -\frac{x}{\left(x^2+6\right)^{3/2}}$

(E) $f'(x) = -\frac{\sqrt{6}x}{\left(x^2+6\right)^{3/2}}$