

§3.2—Rolle's Theorem & the MVT

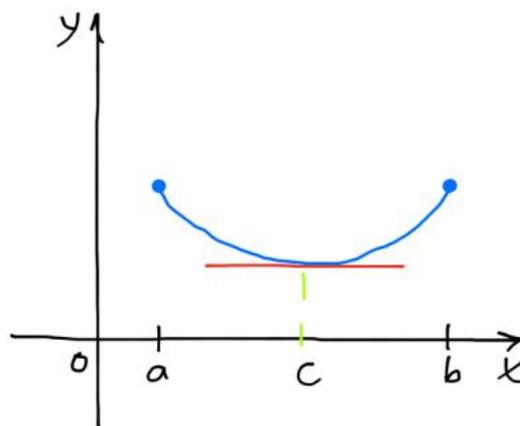
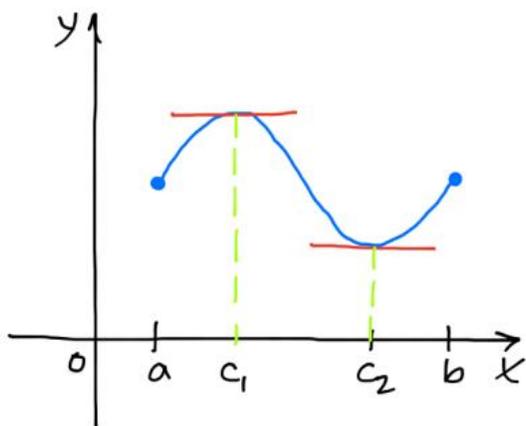
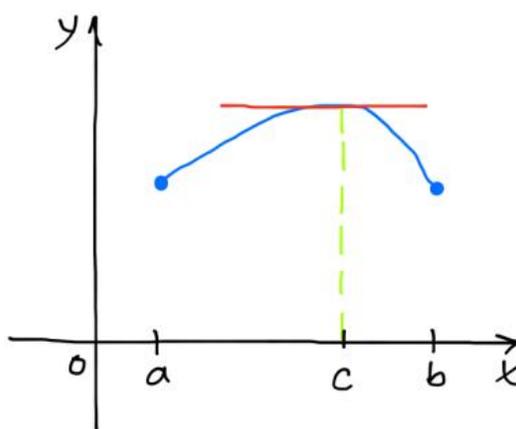
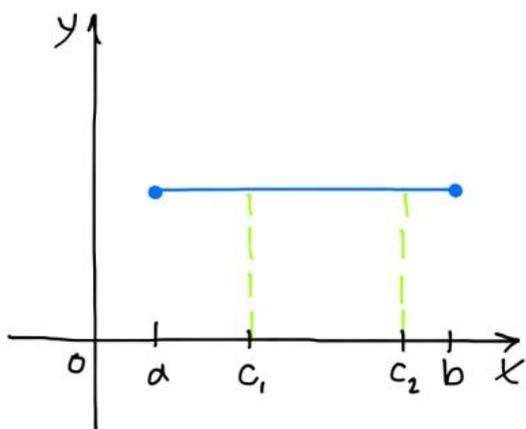
Rolle's Theorem (circa 1691)

Let f be a function that satisfies the following three hypothesis:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b) .
3. $f(a) = f(b)$

Then there is a number $x = c$ in (a, b) such that $f'(c) = 0$

Here are some functions that satisfy all three hypotheses.



As you can see, Monsieur Rolle gave us the necessary conditions in which we are guaranteed to have a horizontal tangent line and a very special critical value. If you think of the position of an object that is thrown straight up from Earth then caught again at the same point, Rolle's theorem guarantees a moment in time when the velocity of that object is zero strictly between the time it was thrown and caught again. Thanks, Rolle!

Example 1:

For $f(x) = x^4 - 2x^2$ on $[-2, 2]$, determine if Rolle's Theorem applies. If so, find the value(s) of $x = c$ guaranteed by the theorem.

Years after Rolle, the preeminent French mathematician Joseph-Louis Lagrange (1736 – 1813) loosened up the qualifications of Rolle's hypothesis, and came up with a much broader, if not more important result.

**The Mean Value Theorem:**

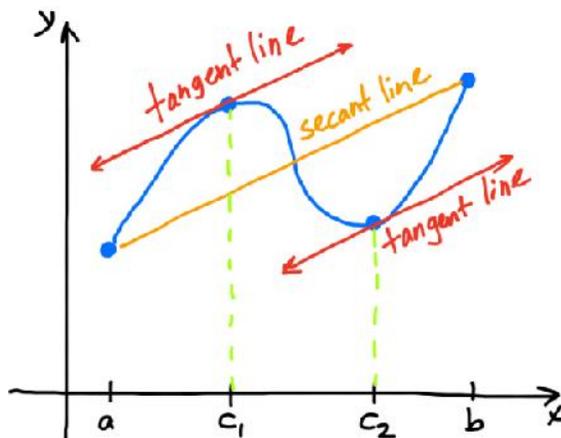
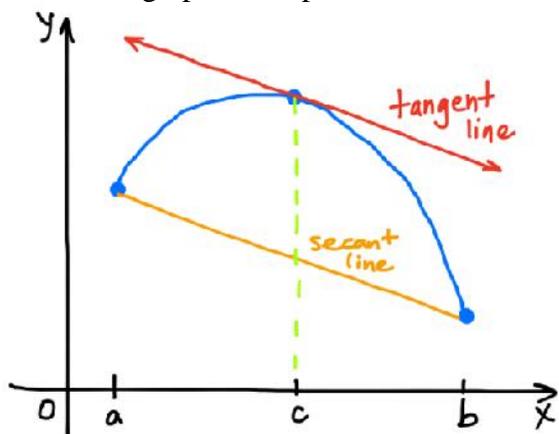
Let f be a function that satisfies the following two hypothesis:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b) .

Then there is a number $x = c$ in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Or $f(b) - f(a) = f'(c)(b - a)$

Here's the graphical implication.



- The slopes of the tangent line(s) and the slope of the secant line on the interval are the same!
- The instantaneous rate of change and the average rate of change on the interval are the same!!

Thanks, Lagrange!!!

Example 2:

Determine if the MVT applies to $f(x) = x^3 - x$ on $[0, 2]$, if so, find the value(s) guaranteed by the theorem.

Example 3:

With the help of your calculator's ability to graphically find zeros, determine all the numbers c which satisfy the conclusion of the Mean Value Theorem for the function $f(x) = x^3 + 2x^2 - x$ on $[-1, 2]$.

It's important to remember the interval in which you're working. Remember that both Rolle's Theorem and the MVT guarantee at least one value strictly on the OPEN interval.

Example 4:

Determine if the MVT applies to $f(x) = x^3 - 3x^2 + 2x$ on the interval $[0, 3]$. If so, find the values guaranteed by the MVT.

Example 5:

For the following functions, determine if the MVT applies. If so, find the value of c guaranteed by the theorem. If not, specifically state why the theorem does not apply.

(a) $f(x) = \frac{x+5}{x-1}$ on $[-3, 5]$

(b) $g(x) = x^{2/3}$ on $[-3, 3]$

Example 6:

The calculus cops have set up their elaborate speed trap on a busy, stretchy stretch of road. A suspected pumpkin farmer who uses the road daily to haul his pumpkins to market is suspected of chronic speeding well beyond the posted limit of 55 mph. The calculus cops aim to finally ticket this unlawful transporter of seasonal gourds. The calculus cops set up 5 miles apart from each other, each parked furtively behind a piece of scenery. Calculus cop A spots the farmer with his jalopy loaded down with would-be jack-o-lanterns. As the farmer passes cop A, he is clocked at a paltry 50 mph. Cop A could have sworn the farmer waves at him as he drives by. Cop A immediately radios calculus cop B 5 miles down the road, whereby cop A starts his timer. Four minutes later, cop B clocks the farmer



cruising by at only 55 mph. He clearly sees the farmer wave at him with a giant grin that would make a jack-o-lantern jealous. After a quick calculation on his field-issued TI-88 ½ calculator, he pulls out with his lights on to issue a speeding ticket to the ornery pumpkin farmer. For what speed can calculus cop B ticket the farmer? Would this ticket hold up in a court of law? Why or why not?

Here's another application of the MVT.

Example 7:

Let $f(x)$ be a function that is differentiable for all x .

(a) Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x . How large can $f(2)$ possibly be?

(b) Suppose that $f(5) = 2$ and $f'(x) \geq -3$ for all x , what is the largest possible value of $f(1)$?