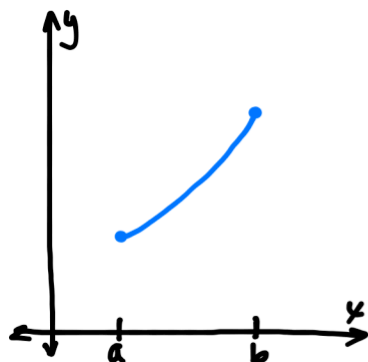


§3.3—Increasing, Decreasing, and the 1st Derivative Test

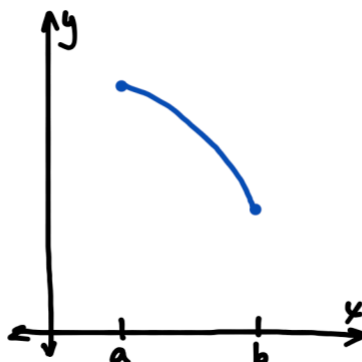
If a graph exists on an interval, it is doing one of three things:

1. Increasing (y -values rise as x -values increase)
2. Decreasing (y -values fall as x -values increase)
3. Staying Constant (y -values stay the same as x -values increase)

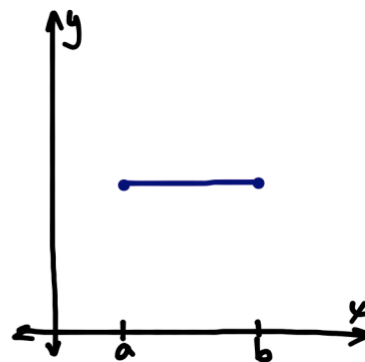
We would like to be able to interpret information about a function f by analyzing information about its derivative $f'(x)$. Since $f'(x)$ tells us the slope of the curve $y = f(x)$ at any point $(x, f(x))$, it tells us whether the curve is going up, going down, or staying the same **at each point**. If we, therefore know the values (or just the **SIGNS**) of $f'(x)$ on a given interval, we know whether the graph is increasing, decreasing, or constant on that interval.



On (a, b) ,
 f is increasing
 \Leftrightarrow
 $f'(x) > 0$



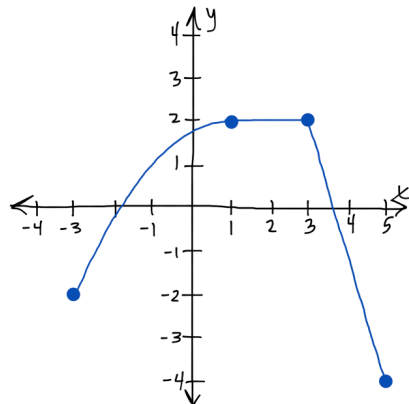
On (a, b) ,
 f is decreasing
 \Leftrightarrow
 $f'(x) < 0$



On (a, b) ,
 f is constant
 \Leftrightarrow
 $f'(x) = 0$

Example 1:

The graph of a function $f(x)$ defined on $[-3, 5]$ is shown. List the **open** intervals over which the function is increasing, decreasing, and/or constant.



$f(x)$ is increasing on

$f(x)$ is decreasing on

$f(x)$ is constant on

Note: A function whose derivative is the same sign on a given interval is said to be **monotonic** on that interval or **strictly** increasing/decreasing/constant on that interval.

We now ask ourselves: At what values of x can the graph of a function change its increasing/decreasing/constant status? OR At what x -values can a function's derivative change signs?

Important Idea

If $f(x)$ is a **continuous function**, then $f'(x)$ can only change its sign at a **critical value**.

If $f(x)$ is a **discontinuous function**, then $f'(x)$ can change its sign either at a **critical value** or a **discontinuity**.

When given the equation of a function $f(x)$, we must find any critical x -values and discontinuities, then **test the intervals** to see what sign f' is in between these x -values using a **number line chart**.

Example 2:

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing and/or decreasing. Justify.

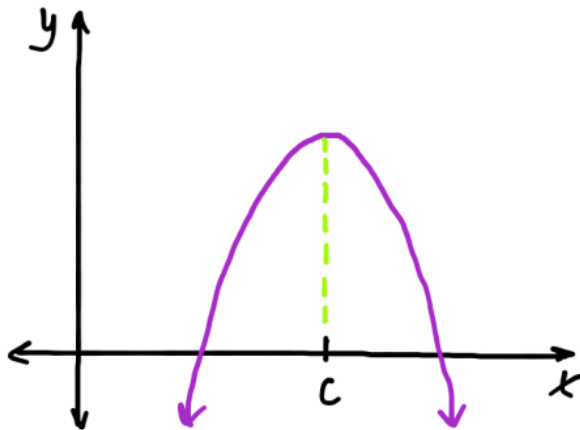
If a continuous function exists at a point, $(c, f(c))$, then knowing that the sign of $f'(x)$ changes signs at that point and **HOW** that sign changes, lends great insight into existence of any Relative Maximums or Relative Minimums.

The First Derivative Test (for Relative Extrema of continuous functions)

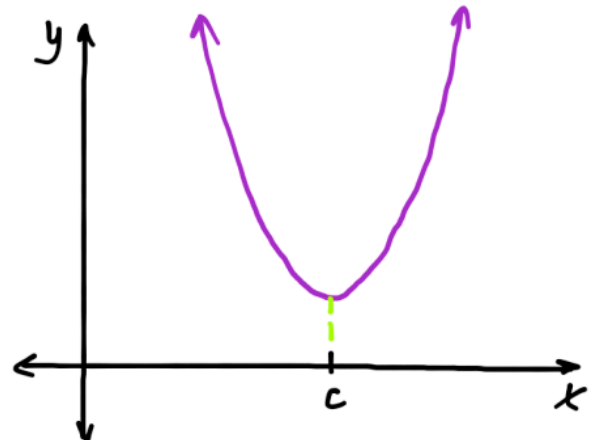
Let $x = c$ be a critical value in the domain of a continuous function $f(x)$, then

1. If $f'(x)$ changes from negative to positive at $x = c$, then f has a **relative minimum** at $x = c$ (or at $(c, f(c))$).
2. If $f'(x)$ changes from positive to negative at $x = c$, then f has a **relative maximum** at $x = c$ (or at $(c, f(c))$).
3. If $f'(x)$ is positive on both sides of $x = c$ or negative on both sides of $x = c$, then $f(c)$ is neither a relative maximum nor a relative minimum.

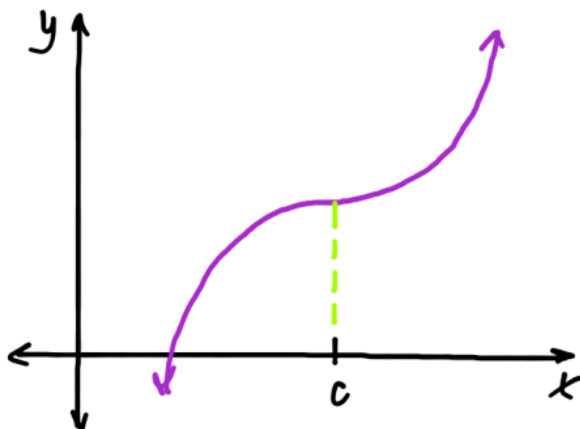
Here's the visualization of the First Derivative Test with justifications. The four graphs below show continuous functions $f(x)$ with critical values $x = c$ marked.



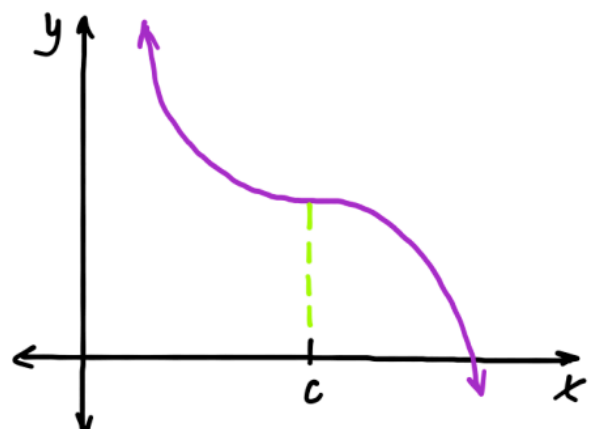
$f(x)$ has a local maximum at $x = c$, because $f'(x)$ changes from positive to negative at $x = c$.



$f(x)$ has a local minimum at $x = c$, because $f'(x)$ changes from negative to positive at $x = c$.



$f(x)$ has neither a local maximum nor a local minimum at $x = c$, because $f'(x)$ is positive immediately on either side of $x = c$.



$f(x)$ has neither a local maximum nor a local minimum at $x = c$, because $f'(x)$ is negative immediately on either side of $x = c$.

NOTE: When justifying a Local Extrema using the First Derivative Test, you MUST write a concluding statement clearly communicating the type of sign change of f' at each $x = c$. Thou mustn't use pronouns either!!

Example 3:

For $f(x) = \frac{2}{5}x^5 - \frac{7}{4}x^4 - \frac{4}{3}x^3 + 9$,

(a) Find the open intervals on which $f(x)$ is increasing and/or decreasing. Justify

(b) Determine the x -values of any local maximums or local minimums of $f(x)$. Justify.

Example 4:

Find the exact values of any relative extrema of the function $f(x) = \frac{1}{2}x - \sin x$ on the interval $[0, 2\pi]$.

Justify.

NOTE: As you noticed in the previous two examples, it is important to answer the question being asked. Are you looking for the x -values where the extrema occur or are you looking for the actual local extrema (y -values) themselves.

Example 5:

Find the relative extrema of $f(x) = (x^2 - 4)^{2/3}$. Justify.

Example 6:

Find the x -coordinates of the relative extrema of each of the following. Justify.

(a) $T(k) = \sqrt[3]{k^2} (2k - 1)$

(b) $J(k) = \sqrt[3]{k} (2k - 1)$

Remember that a local extrema may occur at a discontinuity, as long as the function is defined there. If a function is not continuous at a critical value, the first derivative test cannot be used.

Example 7:

Find the relative extrema of $f(x) = \begin{cases} x+3, & x \neq 1 \\ 2, & x = 1 \end{cases}$. Justify.

Be careful of discontinuities where the function is NOT defined there, especially vertical asymptotes.

Example 8:

For both of the functions below, determine (i) the intervals of increasing/decreasing and (ii) the x -coordinates of the relative extrema. Justify.

$$(a) f(x) = \frac{x^3 + x}{x}$$

$$(b) f(x) = -\frac{x^4 + 1}{x^2}$$