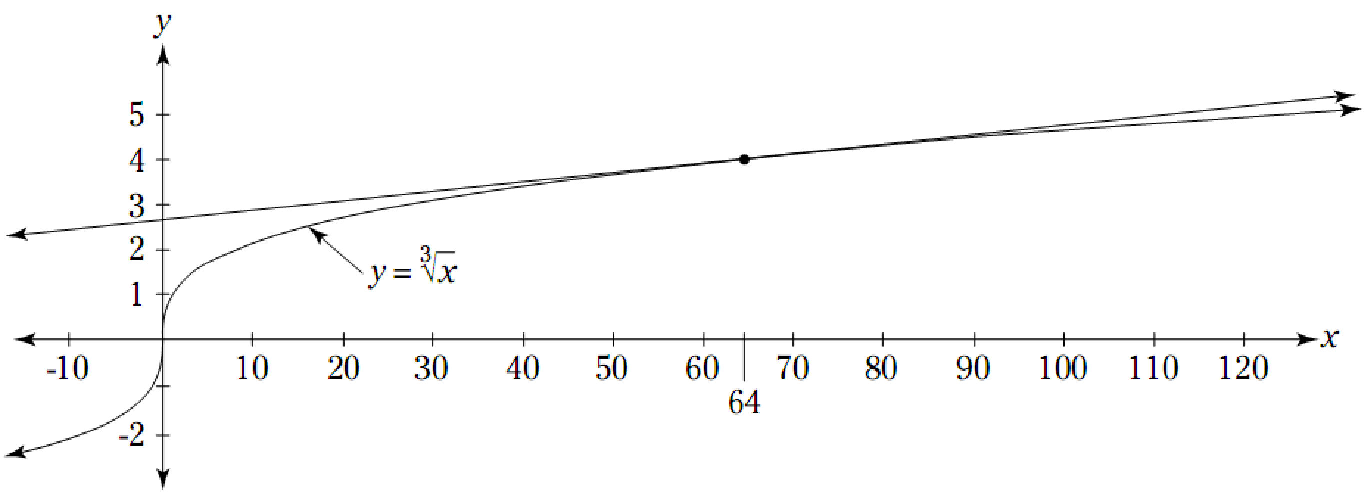
§3.7—Linearization & Differentials

Linear approximation is a very easy thing to do, and once you master it, you can impress all of your friends by calculating things like $\sqrt[3]{70}$ in your head . . . . about 4.125! Impressed? I'll teach you how.

Recall that if a function $f(x)$ is **differentiable** at $x = c$, we say it is **locally linear** at $x = c$. This means that as we zoom in closer and closer and closer and closer around $x = c$, the graph of $f(x)$, regardless of how curvy it is, will begin to look more and more and more and more like the tangent line at $x = c$.

This means that we can use the equation of the tangent line of $f(x)$ at $x = c$ to **approximate** $f(c)$ for values **close to** $x = c$. Let's take a look at $\sqrt[3]{70}$ and the figure below.



Example 1:
Approximate $\sqrt[3]{70}$ by using a tangent line approximation centered at $x = 64$. Determine if this approximation is an over or under-approximation. Approximate $\sqrt[3]{70}$ using a secant line approximation using $x = 64$ and $x = 125$. Determine if this approximation is an over or under-approximation.

How to find linear approximations of $f(x)$ at $x = c$, the center to approximate $f(x)$ at $x = a$, a value near the center $x = c$.

1. Find the equation of the tangent line at the center $\left(c, f(c)\right)$ in point-slope form.
2. Solve for y and rename it $L(x)$.
3. Plug in $x = a$ into $L(x)$ writing the notation VERY CAREFULLY as $f(a) \approx L(a) = \cdots$
4. If asked, determine if $L(a)$ is an over-approximation or an under approximation by examining the concavity of $f(x)$ at the center $x = c$.
   a. If $f''(c) < 0$, $f(x)$ is concave down at $x = c$ then $L(a)$ is an over-approximation
   b. If $f''(c) > 0$, $f(x)$ is concave up at $x = c$ and $L(a)$ is an under-approximation

Example 2:
Estimate the fourth root of 17. Determine if the linearization is and over- or under-approximation.

Example 3:
Approximate 3.01^5. Determine if the linearization is and over- or under-approximation.

Example 4:
Approximate $\ln\left(e^{10} + 5\right)$. Determine if the linearization is and over- or under-approximation.

We'll now introduce a notation and a process that will become second nature to you the rest of the year.

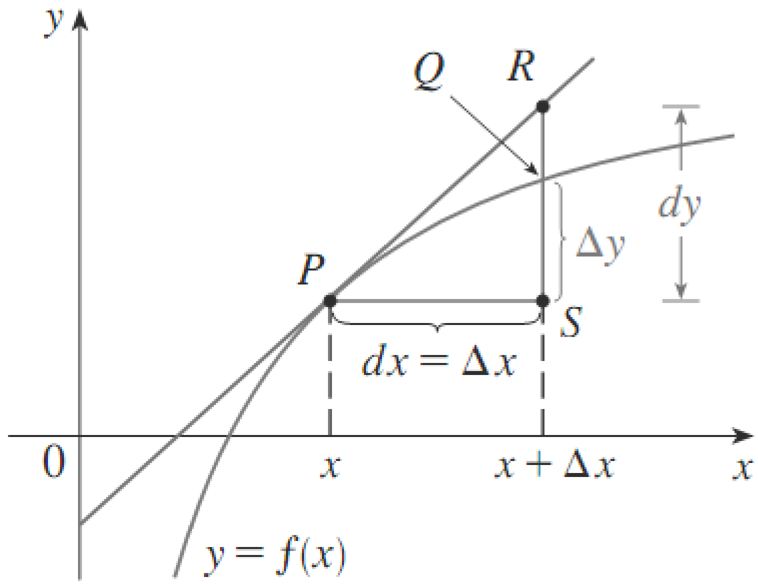
Recall Leibniz's notation for the derivative function: $\frac{dy}{dx} = f'(x)$

dy is called the **differential of *y***, and dx is called the **differential of *x***.

Similar to Δy and Δx, which are finite, measurable quantities, the differentials denote a change in respective values, however, they are infinitely small, immeasurable differences approaching zero.

We can treat them as individual quantities and we can even solve the equation $\frac{dy}{dx} = f'(x)$ for dy:

$$dy = f'(x)dx$$



This is called the "differential form" of the derivative or y. Let's practice on a few.

Example 5:
Find the derivative of each function in differential form for each of the following.

(a) $y = 2t^3 + 5t^2 - 3t + 1$ (b) $z = x^3 \sin(3x)$ (c) $m = e^{5q^2+1}$

Here's a nice application of differentials I think you'll recognize. If Δx is small, we can say that $dy \approx \Delta y$. This is exactly what we did when we did linear approximations! In this new context, though, we can work cooler types of problems like the following.

Example 6:
A machined spherical bearing was measured with a caliper. The bearing's radius was found to be 2.3 inches with a possible error no greater than 0.0001 inches. What is the maximum possible error in the volume of the spherical bearing if we use this measurement for the radius?