

§3.8—Related Rates

Earlier in the year, we used the basic definition of calculus as “the study of change.” Anytime we use any words such as increasing, decreasing, growing, or shrinking, etc., we’re talking about calculus. Change occurs over time, so when we talk about how a quantity changes, we are talking about the derivative of that quantity with respect to time. We can state this mathematically as $\frac{d}{dt}[\text{quantity}]$

Example 1:

Write the following statements mathematically.

- a) Mortimer is growing at the rate of 3 inches per year.

- b) My stock portfolio is losing 5 cents per day.

- c) The radius of a circle gets larger by 4 feet each hour.

- d) The outside temperature is dropping by $5^\circ F$ per minute

Example 2:

The length of a rectangle is decreasing by 2 inches per second and the width is increasing by 3 inches per second. When the length is 10 inches and the width is 6 inches, how fast is the (a) perimeter and (b) area changing?

Example 3:

A right triangle whose sides are changing has sides of 30 and 40 inches at a particular instant. If the shorter of these two sides is increasing at 3 in/sec and the longer side is decreasing at 5 in/sec, how fast is the (a) area and (b) hypotenuse changing?

To solve related rates problems, you need a strategy that always works. Related rates problems always can be recognized by the words “increasing, decreasing, growing, shrinking, changing.” Follow these 11 easy steps in solving a related rates problem.

1. Make a sketch. Label all sides and linear quantities that **CHANGE THROUGHOUT THE LIFE OF THE PROBLEM** as convenient variables of your choice. If something in the problems is **NOT** changing throughout the life of the problem, label it with its **CONSTANT** value.
2. Label all your given rates as $\frac{d[\text{variable}]}{dt}$ with the correct **SIGN**, based on whether that quantity is increasing or decreasing. **If a quantity is increasing, its derivative is positive. If a quantity is decreasing, its derivative is negative.**
3. Identify the “snapshot moment.” Draw these in your camera with retro-flash cube going off (optional, but fun and highly recommended).
4. Identify the rate $\frac{d[\text{variable}]}{dt} = ?$ you’re trying to find. This is your target. Try to hit it.
5. Before you can relate your rates to each other, you need to relate their quantities. Find an equation which ties your variables together. If it an area problem, you need an area equation. If it is a right triangle, the Pythagorean formula may work or general trig formulas may apply. If it is a general triangle, the law of cosines may work. You will need to memorize your geometric formulas. **TRY TO GET CONSTANTS INVOLVED IF YOU CAN!** This equation will be your “position” or quantity equation
6. ****If your equation is a function of two variables (as in the $V = \frac{\pi}{3} r^2 h$, the volume of a right circular cone), you may want to find an equation relating the two variable so that you can write your equation as a function of a single variable, depending on what final rate quantity you’re looking for. This is not always the case. For a Pythagorean Theorem problem, it is easier to leave the equation as a function of two variables.**
7. Differentiate your equation with respect to time. You are doing implicit differentiation with respect to t . This will be your “velocity” or rate equation.
8. Plug in all values for your variables, including your rates. Hopefully, you will know all variables except one. If not, you will need an equation which will solve for unknown variables. Many times, it is the quantity equation from part 5) above. Do this work on the side as to not destroy the momentum of your work so far.
9. Label your answers in terms of the correct units (very important) and be sure you answered the question asked.
10. Write a sentence summarizing your conclusion beginning with the introductory adverbial clause referencing the moment of time. Include units in your sentence and don’t abbreviate.
11. Smile

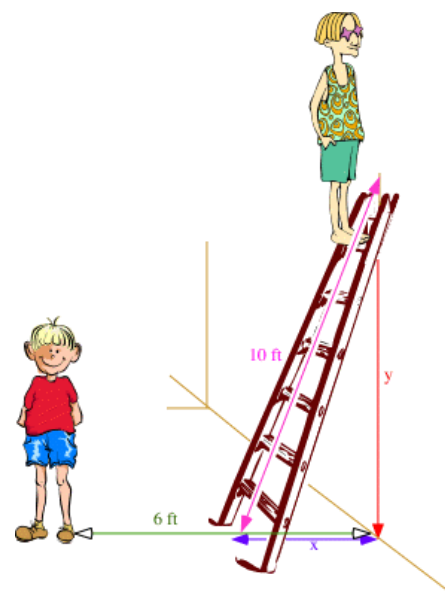
Example 4:

An oil tank spills oil that spreads in a circular pattern whose radius, r , increases at the rate of 50 ft/min. How fast are both the circumference and area of the spill increasing when the radius of the spill is 20 feet. If the depth of the oil spill (yes! Sinking oil!!) is given by $d(r) = 2r^2 + 1$, how fast is the volume of the oil spill changing at this moment?

Example 5:

A 10-foot ladder is leaning against a vertical wall. The lower end of the ladder is being pulled away from the wall at a rate of 2 feet per second. When the lower end of the ladder is 6 feet away from the wall . . .

- How fast is the ladder sliding down the wall?
- How fast is the angle between the ground and the ladder changing?
- How fast is the area of the triangle formed by the ladder and the two walls changing?
- What is the acceleration of the ladder down the wall?



Example 6:

A spherical balloon is being inflated at a constant rate of 5 cubic inches per minute. When the radius of the balloon is 4 inches, how fast is the surface area of the balloon changing?

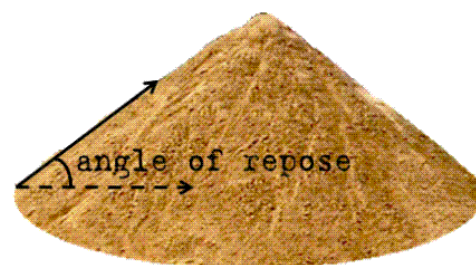
Example 7:

Two cars are riding on two different roads that meet at a 90 degree angle. Car A is 3 miles south of the intersection traveling north at 40 mph, and car B is 2 miles east of the intersection traveling east at 50 mph. How fast is the distance between the two cars changing at this moment?

Example 8:

Sand is being poured onto a beach creating a cone whose base diameter is always twice its height. The sand is being poured at the rate of 20 cubic inches per second. When the height of the conical pile is 6 inches . . .

- how fast is the radius of the pile changing?
- how fast is the diameter of the pile changing?
- how fast is the height of the pile changing?
- If the angle of repose is defined to be the steepest angle at which dry, unconsolidated sediment is stable, what is the sand pile's angle of repose? How fast is it changing?



Example 9:

A Cherry-flavored raspa sno-cone is leaking from its paper cone at a rate of 2 cubic inches per minute. The paper cone's top radius is 2 inches and is 5 inches tall. When the depth of melted cherry raspa mixture is 3 inches,

- how fast is the radius of the raspa changing?
- how fast is the raspa melt leaking onto Niño Korpi's school clothes?
- If Niño Korpi has a small cylindrical cup with a 2 inch diameter beneath the leaking raspa, at this moment, how fast is the height of the raspa juice in the "catch cup" changing?

**Example 10:**

(1984-AB5) The volume V of a cone is increasing at the rate of 28ρ cubic inches per second. At the instant when the radius r on the cone is 3 inches, its volume is 12ρ cubic inches, and the radius is increasing at $\frac{1}{2}$ inches per second.

- At the instant when the radius of the cone is 3 inches, what is the rate of change of the area of the base?
- At the instant when the radius of the cone is 3 inches, what is the rate of change of its height h ?
- At the instant when the radius of the cone is 3 inches, what is the instantaneous rate of change of the area of its base with respect to its height h ?

Example 11

A young boy is out at night, running toward a street lamp at 6 feet per second. If the streetlamp is 30 feet tall and the boy is 5 feet tall in his running stance, when he is 4 feet from the base of the lamppost,

- (a) how fast is his shadow length changing?
- (b) how fast is the tip of his shadow moving?