§4.3—The Fundamental Theorem of Calculus

We've learned two different branches of calculus so far: differentiation and integration. Finding slopes of tangent lines and finding areas under curves seem unrelated, but in fact, they are very closely related. It was Isaac Newton's teacher at Cambridge University, a man name **Isaac Barrow** (1630 – 1677), who discovered that these two processes are actually **inverse operations** of each other in much the same way division and multiplication are. It was Newton and Leibniz who exploited this idea and developed the calculus into it current form.

The lunar crater "Barrow" is named after Isaac Barrow. The Wheel Barrow is not.

The Theorem Barrow discovered that states this inverse relation between differentiation and integration is called **The Fundamental Theorem of Calculus**.

We're now ready for the "shortcut" rule for integration. This is what we've been waiting for: an easier way to calculate definite integrals.

The Fundamental Theorem of Calculus, Part 1 (FTOC1)

If f is continuous on $[a,b]$ and $F(x)$ is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

This integral gives us the NET change!!!

Example 1:
Evaluate the integrals using the FTOC1.

(a) $\int_1^3 \left(e^x - 3\right)dx$ (b) $\int_0^{1/2} \frac{2}{\sqrt{1-x^2}}dx$ (c) $\int_1^{e^2} x^{-1}dx$

Example 2:
Evaluate each definite integral using **any** method. Verify on the calculator.

(a) $\int_1^2 \left(x^2 - 3\right)dx$ (b) $\int_1^4 3\sqrt{x}\,dx$ (c) $\int_0^{\text{®}/4} \sec^2 x\,dx$ (d) $\int_0^2 \left|2x-1\right| dx$

Example 3:

Find the exact value of $\int_{1/2}^{1/e}\left(10x^4-2\left(1-x^2\right)^{-1/2}-\frac{5}{x}\right)dx$

Just as we can use areas of regions to help us find definite integral values, now that we (officially) know how to evaluate definite integrals, we can use integrals to help us find areas. But we must use caution, for area is always POSITIVE, and WE are responsible for making negative regions positive!!

In such cases, we must CLEARLY IDENTIFY THE INDICATED REGION!

Example 4:

Find the area bounded by the parabola $y=x^2-1$ and $y=0$ from $x=0$ to $x=3$

(a) Without a calculator (b) With a calculator

Example 5:

Find the area of the region bounded by the curves $y=0$, $y=\frac{2}{x}-1$, $x=1$ and $x=e$

(a) Without a calculator (b) With a calculator

Example 6:

Without a calculator, find the area of the region bounded by the *x*-axis and the function $y=\frac{-20}{x^2+1}$ on the interval $-1\le x\le 1$. Use the symmetry of the function to help you evaluate the integral.

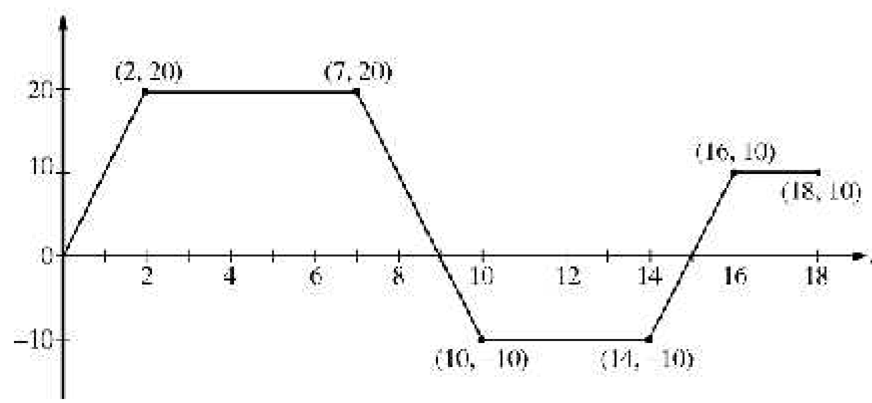
The second part of the theorem deals with integral equations of the form

$$F(x)=\int_a^x f(t)\,dt$$

where *f* is a continuous function on $[a,b]$, and *x* varies between *a* and *b*. Notice that this integral equation is a function of *x*, which appears as the upper limit of integration. If $f(t)$ happens to be positive, and we let $x\in(a,b]$, then we can define $F(x)$ as the area under the curve from *a* to *x*.

Example 7:

The graph of $f(t)$ is given. Let $F(x)=\int_2^x f(t)\,dt$. Use the areas of the regions to find the following:



(a) $F(2)$

(b) $F(9)$

(c) $F(15)$

(d) $F(18)$

(e) $F(0)$

Example 8:

If $F(x)=\int_3^x \left(t^2+t+1\right)dt$, find a simplified, expanded version of $F(x)$ by evaluating the definite integral.

Once you find $F(x)$, find its derivative, $F'(x)$. What do you notice?

The Fundamental Theorem of Calculus Part 2 (FTOC2)—special case

If *f* is a continuous function on $[a,b]$, then the function *F* defined by

$$F(x)=\int_a^x f(t)\,dt,\ a\le x\le b$$

is continuous on $[a,b]$ and differentiable on (a,b). Additionally, $F'(x)=f(x)$. We can also say that

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]=f(x)$$

Example 9:

Evaluate (a) $\frac{d}{dx}\left[\int_1^x \sec t\ dt\right]=$ (b) $\frac{d}{dx}\left[\int_5^x \sqrt{p^3}\sin p\ dp\right]=$ (c) $\frac{d}{dx}\left[\int_{\text{®}}^x \frac{e^k\sqrt{k+k^2}}{\ln k}\ dk\right]=$

Chain Rule anyone?

The FTOC2, most general form:

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)dt\right] = f\left(g(x)\right)\cdot g'(x) - f\left(h(x)\right)\cdot h'(x)$$

Example 10:
Evaluate the following using the FTOC2, then verify by doing in the Looooooong way.

(a) $\frac{d}{dx}\left[\int_{1}^{2x^3} \sec^2 t\ dt\right]$

(b) $\frac{d}{dx}\left[\int_{e^x}^{7} \left(t^2+5t\right)dt\right]$

Example 11:

If $F(x) = \int_{2+\sin 2x}^{3^x} \ln t\ dt$, find $F'(x)$

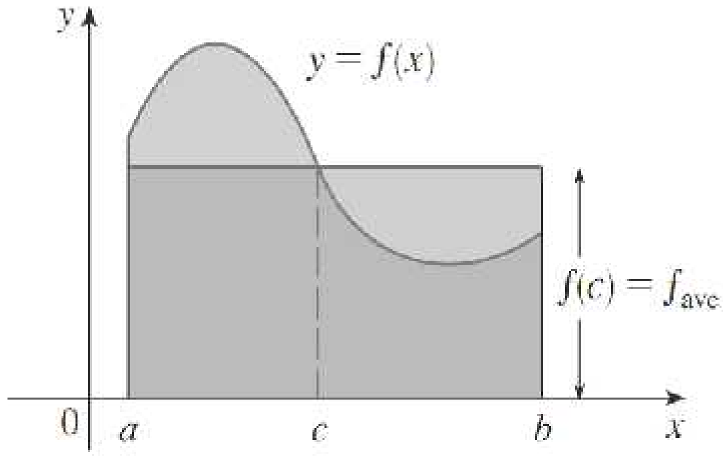
The Mean Value Theorem (for Integrals)

If f is continuous on the closed interval $[a,b]$, then there exists a number $x = c$ in the CLOSED interval $[a,b]$ such that

$$\int_a^b f(x)dx = f(c)\cdot(b-a)$$

Where $f(c)$ is called the **average value** of the function f on the interval $[a,b]$. The above equation above can be explicitly solved for $f(c)$.

$$f\left(c\right)=\frac{\int_a^b f\left(x\right)dx}{b-a} \quad \text{or} \quad f\left(c\right)=\frac{1}{b-a}\int_a^b f\left(x\right)dx$$



Example 12:
(Calculator) In New Braunfels, the temperature (in $^\circ F$) t hours after 9 a.m. was modeled by the function $T\left(t\right)=50+14\sin\frac{\text{®}t}{12}$. (a) Find the average temperature during the 12-hour period from 9 a.m. to 9 p.m. **<u>Show your integral set up</u>**! (b) find the value(s) of t at which the MVT guarantees the temperature in N.B. was the average temperature.

Example 13:
Find the value of c guaranteed by the MVT for integrals for $f\left(x\right)=1+x^2$ on $\left[-1,2\right]$. Interpret the result graphically.

Example 14:
Show that the average rate of change of a car's position over a time interval $\left[t_1,t_2\right]$ is the same as the average value of its velocity function over the same interval.

Example 15:
The table below gives values for a continuous function. Using the values given, find the arithmetic mean of $f(x)$. Using a trapezoidal approximation using 6 subintervals, estimate the average value of f on $[20, 50]$. For this continuous function, is it possible to determine which is more accurate? Is one method better practice than the other?

x	20	25	30	35	40	45	50
$f(x)$	42	38	31	29	35	48	60

Sometimes we might have to solve an integral equation! Being able to simplify definite integrals with variables in the interval of integration is important. Here are a couple of examples showing an important application that is important.

Example 16:
Find the number(s) b such that the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal to 3.

Example 17:
Solve the following conditional equations for the indicated variable.

(a) $\int_{-5}^{x} \left(t^3 - 7t\right) dt = 0$, for x

(b) $\int_{-3}^{k} \left(x^2 - 4\right) dx = 0$, for k

IMPORTANT Example 18:
If $f'(x)=2x^2-2$ and $f(0)=5$, find $f(2)$ by (a) finding the particular solution to the differential equation, then evaluating the solution at $x=2$, and then by (b) using a definite integral.

In the previous example, the first method relied heavily upon our ability to find the antiderivative of the integrand. **<u>This is not always easy, possible, or prudent!</u>** Being able to express a particular value of a particular solution to a derivative as a definite integral is of paramount importance, especially when we don't know how to find a general antiderivative.

Hard Facts To Refute:

A. *Where you are at any given time is a function of 1) where you started and 2) where you've gone from your starting point (displacement).*

B. *What you have at any given moment is a function of 1) what you started with plus 2) what you've accumulated since then.*

When you accumulate at a variable rate, you can to use the definite integral to find your **net accumulation**.

Important Idea of Accumulation********************************(* means VERY IMPORTANT)

What I have now = What I started with + What I've accumulated since I started

This can be expressed mathematically as

$$f(b)=f(a)+\int_a^b f'(x)\,dx$$

Example 19:
If $f'(x) = 4\sin^2(2x)$ and $f(2) = -2$, using your calculator, (a) through (c) only, find the following. Be sure to show your INTEGRAL SET UP for parts (a) through (c).

(a) $f(3)$ (b) $f(5)$ (c) $f(-2)$ (d) an integral equation for $f(x)$

Example 20:
(Calculator permitted) Raw sewage is **leaking** from a storage tank at a rate given by the equation $S(t) = e^{\sqrt{t}}$, where S is measured in gallons per hour, and t is measured in hours. If at $t = 2$ hours there were 572 gallons of raw sewage in the tank,

(a) How much sewage has leaked out of the tank from $t = 2$ hours to $t = 5$ hours?

(b) How much sewage was initially in the tank when it started leaking at $t = 0$ hours?

(c) Write an integral equation that gives the amount of raw sewage in the tank, $A(t)$, at time *t*.

Example 21:
(Calculator Permitted) At the start of Christmas Break, at $t = 0$ days, a man weighed 180 pounds. If the man gained weight during the break (I'm not mentioning any names . . . or even saying that it's me . . . OK, I have a weakness for fruit cake . . .) at a rate modeled by the function $W'(t) = 10\sin\left(\frac{®t}{8}\right)$ pounds per day,

(a) What was the man's weight (in pounds) at the end of the break, 14 days later? Show your integral set up, and don't judge.

(b) At what time was the man at his heaviest during the break? What was his weight at this time? Impressive? Show the ~~food~~ work that leads to your answer.