§5.1—Separable Differential Equations

A **differential equation** is an equation that has one or more derivatives in it. The order of a differential equation is the highest derivative present in the equation.

We know that antidifferentiation, indefinite integration, and solving differential equations all imply the same process. The differential equations we've seen so far have been explicit functions of a single variable, like $\frac{dy}{dx}=3x^3+4x$ (1st order) or $f'(x)=\sin x$ (1st order) or even $h''(t)=5t-\frac{1}{t^2}$ (second order). Solving these equations meant getting back to either $y=$ or $f(x)=$ or $h(t)=$. The general solution meant $+C$. The particular solution required an initial condition and meant we had to find C.

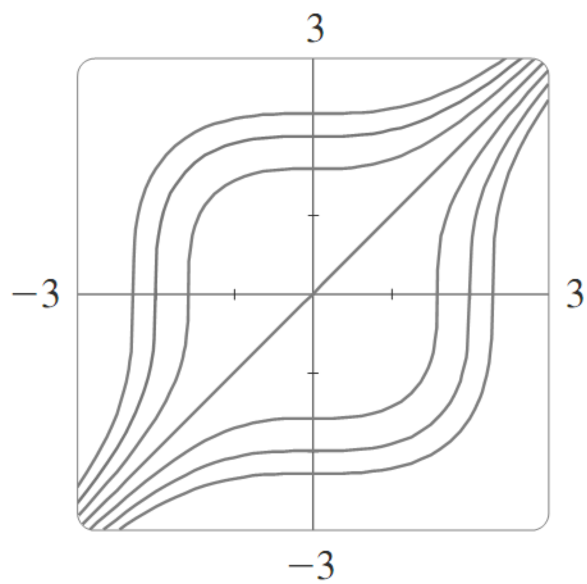
A **separable differential equation** is one in which all x and dx's can be separated from all the y and dy's.

A **first-order separable differential** can be written in the form $\frac{dy}{dx}=f(x)g(y)$, that is, as a ***PRODUCT*** of x-factors and y-factors.

As you will see in the next section, this separation is not always easy or possible; however, in this section we will focus on developing analytic methods for solving 1st order, separable differential equations.

For these types of problems, it is very, very, very important to SHOW THE SEPARATION OF THE VARIABLES.

Example 1:
The graph of several solutions to the differential equation $\frac{dy}{dx}=\frac{x^2}{y^2}$ is shown. Solve the equation, then find the particular solution that satisfy the initial conditions (a) $y(0)=2$, (b) $y(0)=-2$, and (c) $y(0)=0$.



Example 2:

Find the general and particular solutions to the separable differential equation $\frac{dy}{dx} = x^2 y$ given the initial conditions (a) $f(0) = 1$ and (b) $f(0) = -2$.

Example 3:

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Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f, the slope is given by $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.

(b) Write an equation of the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.

(c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$ with the initial condition $f(1) = 4$.

(d) Use your solution from part (c) to find the exact value of $f(1.2)$.

(e) What would the solution equation be if the initial condition were $f(1) = -4$?

Example 4:

For each of the following, find the general solution $y = f(x)$ by algebraically manipulating the differential equation first to the separable form $\frac{dy}{dx} = f(x)g(y)$

(a) $\frac{dy}{dx} = e^{x-y}$

(b) $\frac{dy}{dx} - x = xy^2$

Example 5:

For the differential equation $y\frac{dy}{dx} - x = 0$

(a) Find the general solution equation.

(b) Find the particular solution that pass through $(0,-4)$.

Law of Exponential Change

For exponential growth functions, the more you have, the more you get. For exponential decay functions, the less you have, the less you lose. Quantities that grow/decrease by a factor or a percentage at regular intervals, are exponential. This can be stated equivalently as:

The rate of change of a quantity is directly proportional to that quantity itself.

Mathematically, we state this as $\frac{dy}{dt} = ky$, where k is either a growth ($k > 0$) or decay ($k < 0$) constant.

Example 6:

Solve the separable differential equation $\frac{dy}{dt} = ky$

MEMORIZE. MEMORIZE. MEMORIZE.

If $\frac{dy}{dt} = ky$, then $y = Ce^{kt}$, where C is the initial amount present (*y*-intercept of the graph).

Example 7:

The population of bacteria in a culture increased from 400 to 1600 in three hours. Assuming that the Bacteria population, *P*, grows according to the rate equation $\frac{dP}{dt} = kP$, where *t* is the time in hours.

(a) Find the value of k

(b) How fast is the population of bacteria increasing when the population is 3000?

Example 8:

An amount, A, increases at a rate proportional to its current amount, such that $\frac{dA}{dt} = kA$, where t is measured in hours. If the amount triples every 11 hours, find the value of k (both exact and 3-decimal approximation).

Example 9:

The weight of a population of bacteria is given by a differentiable function B, where $B(t)$ is measured in tons and t is measured in hours. The weight of the bacteria population increases according to the equation $\frac{dB}{dt} = kB$, where k is a constant. At time $t = 0$, the weight of the bacteria population is 3 tons and is increasing at a rate of $\frac{2}{5}$ tons per hour. What is an expression for $B(t)$?

Example 10:

Radium-226 ($^{226}_{88}Ra$) loses its mass at a rate that is directly proportional to its mass. If its half-life is 1590 years, and if we start with a sample of radium-226 with a mass of 100 mg,

(a) Find the formula for the mass, $M(t)$ that remains after t years.

(b) How many mg of the original sample remains after 100 years?

(c) How many years (exact answer) will it take for the sample to have only 3 mg remaining?

Example 11:
Suppose the amount of oil pumped from one of the canyon wells in Whittier, California, decreases at the continuous rate of 10% per year. After how many years will the well's output fall to one-fifth of its present level?

Example 12:
Suppose that a population of fruit flies grows in proportion to the number of fruit flies in the population. If there were 100 flies after the second day and 300 flies after the fourth day, how many flies were in the original population?

Finish Strong!

Example 13:
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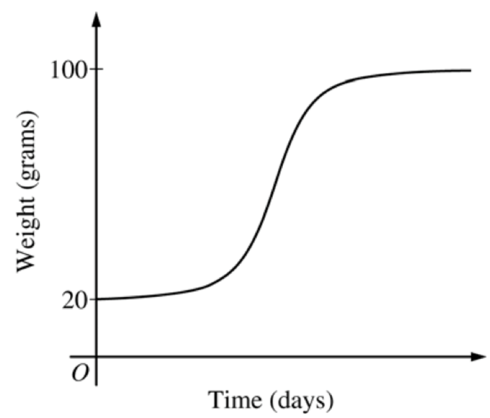
The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t=0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time *t* days after it is first weighed, then

$$\frac{dB}{dt}=\frac{1}{5}(100-B).$$

Let $y=B(t)$ be the solution to the differential equation above with initial condition $B(0)=20$.

(a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.

(b) Find $\frac{d^2B}{dt^2}$ in terms of *B*. Use $\frac{d^2B}{dt^2}$ to explain whey the graph of *B* cannot resemble the following graph.



(c) Use separation of variables to find $y=B(t)$, the particular solution to the differential equation with initial condition $B(0)=20$.