§5.4—Integration by Parts

In the best movie of all time about a high school calculus teacher, Stand and Deliver, Edward James Olmos, portraying Jaime Escalante, says, "Calculus is not meant to be made easy, it already is."

He also says, in my favorite scene in Silver Screen history, "Do you want me to do [the math problem] for you?" (student quickly says yes) "YOU'RE **SUPPOSED** TO SAY '**NO'**!". . . "It's as easy as Tic . . . Tac . . . Toe."

Olmos's character was referring to two things: Ganas and Integration by Parts.

The problem Tito was trying to do in the movie was the following:

$$\int x^2 \sin x dx$$

This integral doesn't fit any of our patterns so far, nor can we, through clever algebraic manipulation, get it into one of our recognizable patterns. There is, in fact, a new pattern we want to take notice of, and it is this.

We have two factors. We know how to differentiate one (or both) of them and we know how to integrate the other one (or both) of them.

When you see this pattern, it's time for integration by parts. We'll start with an example easier than the one from the movie. We'll also talk about backwards Zorro.

Example 1:
Evaluate $\int x \cos x dx$

Why the heck does that work?

Remember the product rule of differentiation (don't answer that if you don't).

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

Rearranging and integrating both sides, we get:

$$\int f'(x)g(x)dx + \int f(x)g'(x)dx = \int \frac{d}{dx}\left[f(x)g(x)\right]dx$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

If we, for the sake of simplicity, let $u = f(x)$ and $v = g(x)$, we get

$$\boxed{\int udv = uv - \int vdu}$$

Example 2:
Evaluate $\int 2xe^{4x}dx$

Example 3:
Evaluate $\int x\ln x dx$

Example 4:
Evaluate $\int \ln x dx$

Example 5:
Evaluate $\int \arctan x dx$

Example 6:
Evaluate $\int x \arctan x dx$

Example 7:
Evaluate $\int \theta \sec^{-1} \theta d\theta$

Let's revisit the one from the movies:

Example 8:
Evaluate $\int x^2 \sin x dx$. In the movie, Tito was using the "backwards Zorro" method, and chose $u = \sin x$ and $dv = x^2 dx$. Use Tito's method and feel his frustration. Learn from it.

The "Tic-Tac-Toe" a.k.a. the Tabular Method
In general, we like to choose our polynomial term as *u*. The exception would be if there is a log or inverse trig factor. When our polynomial is degree two or higher, we can abandon the "repeated backwards Zorro" method for the Tabular method. It's as easy as Tic-Tac-Toe.

Example 9:
Evaluate $\int x^2 \sin x dx$ using the tabular method.

Example 10:

Evaluate $\int t^4 e^{2t}\,dt$

Example 11:

Solve $\frac{dz}{dx} = x^3 \ln x$, if $z = 5$ when $x = 1$

Example 12:

If $f(x) = x^3 \cos 2x$, find the average value of f on the interval $\left[0, \frac{\pi}{2}\right]$. Verify on your calculator.

Example 13:

Evaluate $\int \frac{e^x}{\sec x}\,dx$