§6.1—Integral as Net Change

Recall that the definite integral gives us the Net Accumulation over an interval. For things that change, we can use the definite integral to model a myriad of real-world applications.

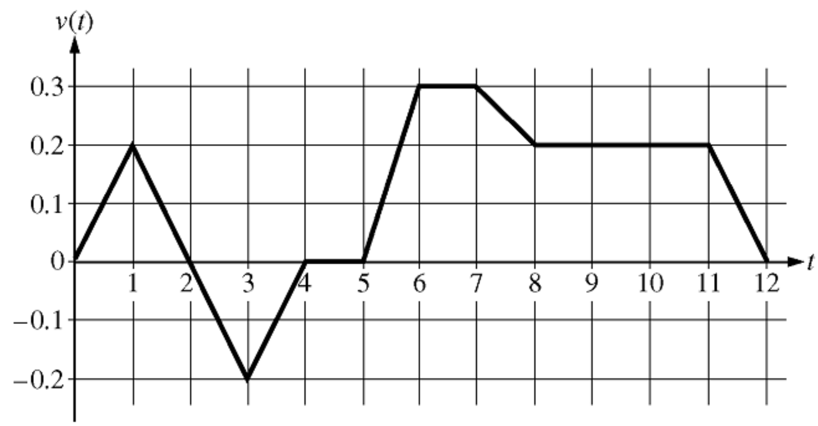
Example 1:
A car travels at a constant velocity of 60 miles per hour in one direction along a straight road for two hours. After two hours, how far has the car traveled? Graph the car's velocity against time. How does your answer manifest itself with respect to the graph?

Example 2:
Suppose a car travels along a straight road according to the velocity function $v(t)=t^2-4$, for $t\in[0,5]$. Where *v* is measured in miles per hour and *t* is in hours. Sketch the velocity of the function over the interval. During what times is the car traveling in the negative direction? Positive direction? At what time does the car change directions? Evaluate $\int_0^2 v(t)\,dt$, $\int_2^5 v(t)\,dt$, $\int_0^5 v(t)\,dt$, and $\int_2^0 v(t)\,dt+\int_2^5 v(t)\,dt$ and interpret what each of them means in the context of the car. Include units in your analysis. Suppose the car started 4 miles away from a checkpoint (on the positive side) initially. Where is the car, in relation to the check point, at $t=5$

Motion Summary:

- Displacement = Net Change in Position = $\int_a^b v(t)dt$
- Distance Traveled = Gross Distance Traveled = $\int_a^b \text{speed}dx = \int_a^b |v(t)|dt$
- Position (at $t = b$) = $s(b) = s(a) + \int_a^b v(t)dt$

Example 3:
AP 2009-1 (Calculator Permitted)
Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.



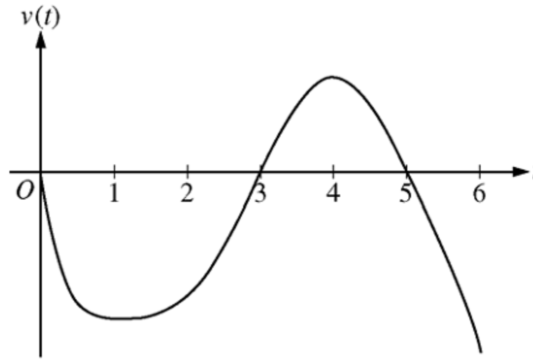
a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|dt$.

c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. AT what time does she turn around to go back home? Give a reason for your answer.

d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to the school: Caren or Larry? Show the work that leads to your answer.

Example 4:
AP 2008-4 (No Calculator)



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0,3]$, $[3,5]$, and $[5,6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

Although accumulating velocities and distances is a very important application of the integral, we can accumulate oh so many other things.

Here's the basic premise: If you have a rate equation that describes how something changes, the integral of that rate equation over an interval gives you the net accumulation of that something. This brings us back to this:

What you have at any given moment is a combination of what you started with plus what you've accumulated since then.

Example 5:
AP 2010-2 (Calculator permitted)

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon ($t = 0$) and 8 P.M. ($t = 8$). The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table.

a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

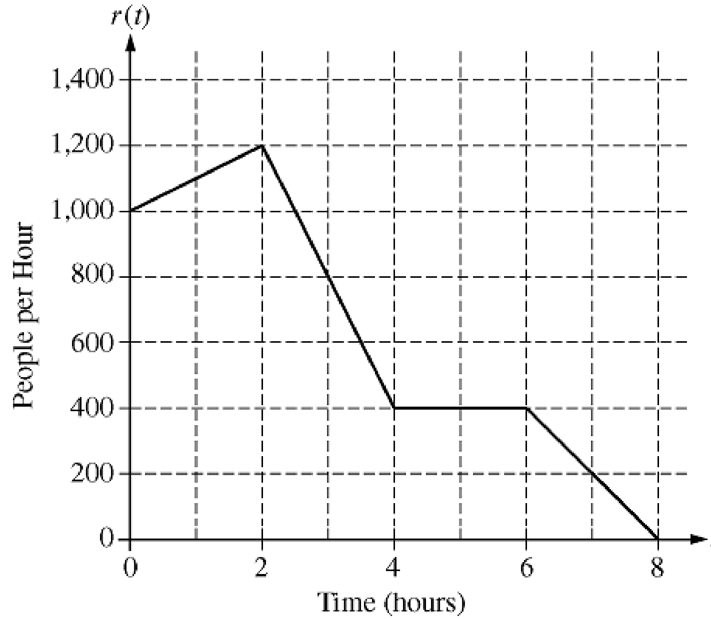
b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

c) AT 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight ($t = 12$)?

d) According to the model from part c), at what time were the entries being processed most quickly? Justify you answer.

Sometimes you are gaining while your are losing. Think of pouring water into a bucket that has a small hole at the bottom. In this case . . .

What you have at any given moment is a combination of what you started with plus what you've accumulated since then minus how much you have lost since then.



Example 6:
AP 2010-3 (Calculator Permitted)
There are 700 people in line for a popular amusement-park ride when the ride begins operation in the morning. Once it begins operation, the ride accepts passengers until the park closes 8 hours later. While there is a line, people move onto the ride at a rate of 800 people per hour. The graph shows the rate, $r(t)$, at which people arrive at the ride throughout the day. Time t is measured in hours from the time the ride begins operation.

a) How many people arrive at the ride between $t=0$ and $t=3$? Show the computations that lead to your answer.
b) Is the number of people waiting in line to get on the ride increasing or decreasing between $t=2$ and $t=3$? Justify your answer.
c) At what time t is the line for the ride the longest? How many people are in line at that time? Justify your answers.
d) Write, but do not solve, an equation involving an integral expression of r whose solution gives the earliest time t at which there is no longer a line for the ride.

Sometime we variable rates of accumulation that vary within and between time intervals—piecewise anyone?

Example 7:
AP 2010-1 (Calculator Permitted)

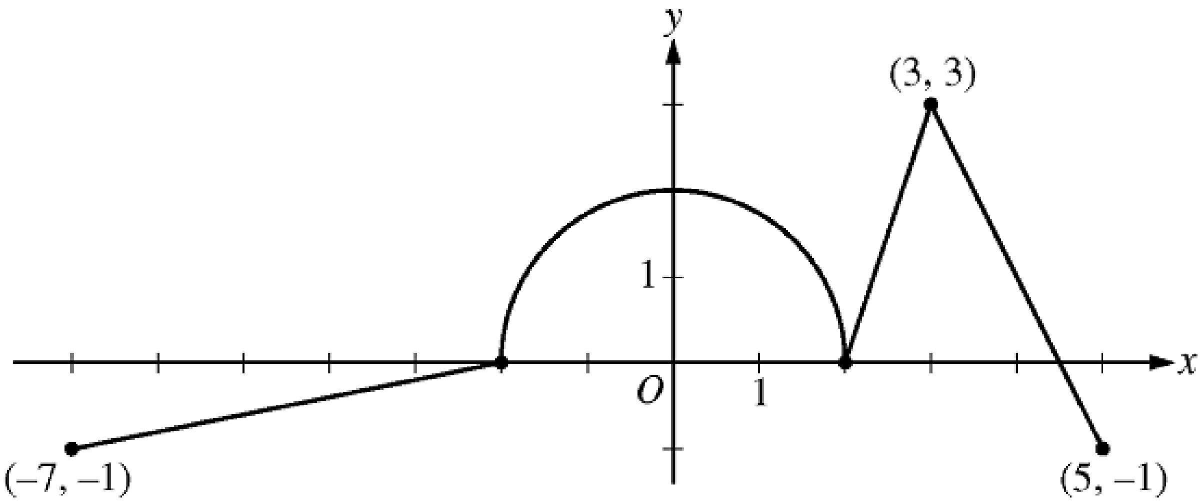
There is no snow on Janet's driveway when snow begins to fall at midnight. From midnight to 9 A.M., snow accumulates on the driveway at a rate modeled by $f(t) = 7te^{\cos t}$ cubic feet per hour, where t is measured in hours since midnight. Janet starts removing snow at 6 A.M. ($t = 6$). The rate $g(t)$, in cubic feet per hour, at which Janet removes snow from the driveway at time t hours after midnight is modeled by

$$g(t) = \begin{cases} 0 & \text{for } 0 \le t < 6 \\ 125 & \text{for } 6 \le t < 7 \\ 108 & \text{for } 7 \le t \le 9 \end{cases}$$

a) How many cubic feet of snow have accumulated on the driveway by 6 A.M.?
b) Find the rate of change of the volume of snow on the driveway at 8 A.M.
c) Let $h(t)$ represent the total amount of snow, in cubic feet, that Janet has removed from the driveway at time t hours after midnight. Express h as a piecewise-defined function with domain $0 \le t \le 9$.
d) How many cubic feet of snow are on the driveway at 9 A.M.?

Sometimes we just accumulate y-values and no units are involved.

Example 8:
AP 2010-5 (No Calculator)



Graph of g'

The function g is defined and differentiable on the closed interval $[-7,5]$ and satisfies $g(0)=5$. The graph of $y=g'(x)$, the derivative of g, consists of a semicircle and three line segments, as shown in the figure above.

a) Find $g(3)$ and $g(-2)$.

b) Find the x-coordinate of each point of inflection of the graph of $y=g(x)$ on the interval $-7<x<5$. Explain your reasoning.

c) The function h is defined by $h(x)=g(x)-\frac{1}{2}x^2$. Find the x-coordinate of each critical point of h, where $-7<x<5$, and classify each critical point as the location of a relative minimum, relative maximum, or neither a minimum nor a maximum. Explain your reasoning.