§6.2—Areas between Curves

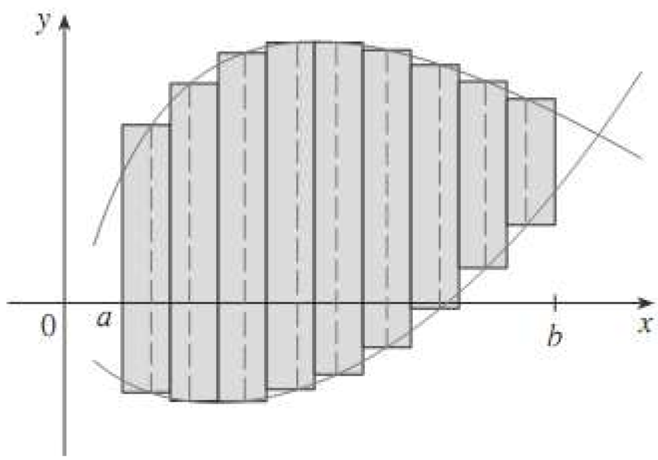
Area is always positive. Up to now, we've only considered area between a curve and the *x*-axis. We now look at a way to find the area of a region bounded by two or more curves.

Repeat the following several times: **"Top minus Bottom or Right minus Left"**

We first learned to approximate areas by using rectangular approximations. You can think of each of those rectangles as being slices of the region. The sum of the areas of the slices give you an approximation of the area of the region.

If we can find a way to represent the height of each of those **representative rectangular** slices as a positive expression, we can integrate through a uniformly, infinitely small width.

Here's what the slices might look like between two curves *f* and *g* on an interval $[a,b]$.



The height of a representative rectangle on this interval, as a function of *x*, is given by $h(x)= f(x)-g(x)$ or $h(x)=TOP-BOTTOM$. The Area of the region can the be expressed as a single integral:

$$A=\int_a^b \left[f(x)-g(x)\right]dx$$

The steps to a successful problem involving the area between two curves are to:

- draw a picture
- clearly identify the region
- decide how to slice the region—vertically or horizontally
- find/identify intervals of integration
- set up the equation for the height of a representative rectangle.

Example 1:

(No Calculator) Find the area of the region bounded by $f(x) = e^x$, $g(x) = x$, and by the vertical lines $x = 0$ and $x = 1$. Verify on your calculator and compare answers.

Example 2:

(No Calculator) Find the area of the region enclosed by $y = x^2$ and $y = 2x - x^2$.

Example 3:

(Calculator Permitted) Find the area of the region bounded by $f(x)=\dfrac{x}{\sqrt{x^2+1}}$ and $g(x)=x^4-x$.

Example 4:

(Calculator Permitted) Find the area of the region between the graphs of $f(x)=3x^3-x^2-10x$ and $g(x)=2x-x^2$.

Example 5:
(No Calculator) Using a vertical slicing method, find the area of the region in the first quadrant bounded by $y=\sqrt{x}$, $y=0$, and $y=x-2$

Example 6:
(No Calculator) Using a horizontal slicing method, find the area of the region in the first quadrant bounded by $y=\sqrt{x}$, $y=0$, and $y=x-2$

Example 7:

(Calculator Permitted) Find the area of the region enclosed by the graphs of $y = x^3$ and $x = y^2 - 2$

Example 8:

(Calculator Permitted) Given $y = e^x + 1$ and $y = 2 - x^2$

(a) Sketch the graphs of the two functions, and identify the region bounded by the two graphs.

(b) Find the coordinates (x, y) of the points of intersection of the two graphs. Show the work that leads to your answer.

(c) Using **<u>vertical slices</u>**, set up and evaluate an integral that gives the area of the region bounded by the two graphs (3 decimal accuracy).

(d) Using **<u>horizontal slices</u>**, set up and evaluate an integral that gives the area of the region bounded by the two graphs (3 decimal accuracy).

Example 9:
Similar to AP MC Question

Which of the following limits is equal to $\int_1^6 x^3\,dx$?

(A) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^3\left(\frac{1}{n}\right)$

(B) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)^3\left(\frac{5}{n}\right)$

(C) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{5k}{n}\right)^3\left(\frac{1}{n}\right)$

(D) $\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{5k}{n}\right)^3\left(\frac{5}{n}\right)$