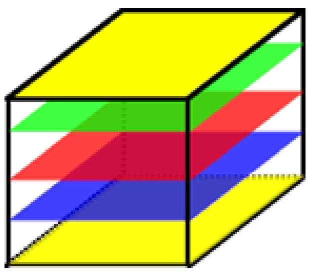
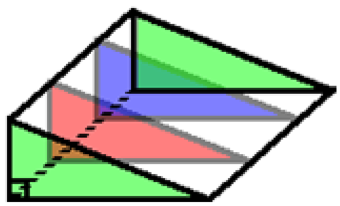
§6.3—Volumes

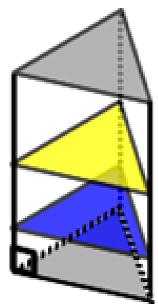
Just as area is always positive, so is volume and our attitudes towards finding it.

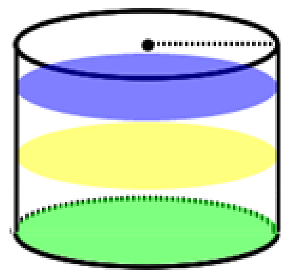
Let's review how to find the volume of a regular geometric prism, that is, a 3-dimensional object with two regular faces separated by some distance, *h*. Whether it is a rectangular prism, triangular prism, or a circular prism (cylinder), etc., if we can find the area of the face, we need only multiply by the distance between the two faces, *h*. In general

$$V = A \cdot h$$





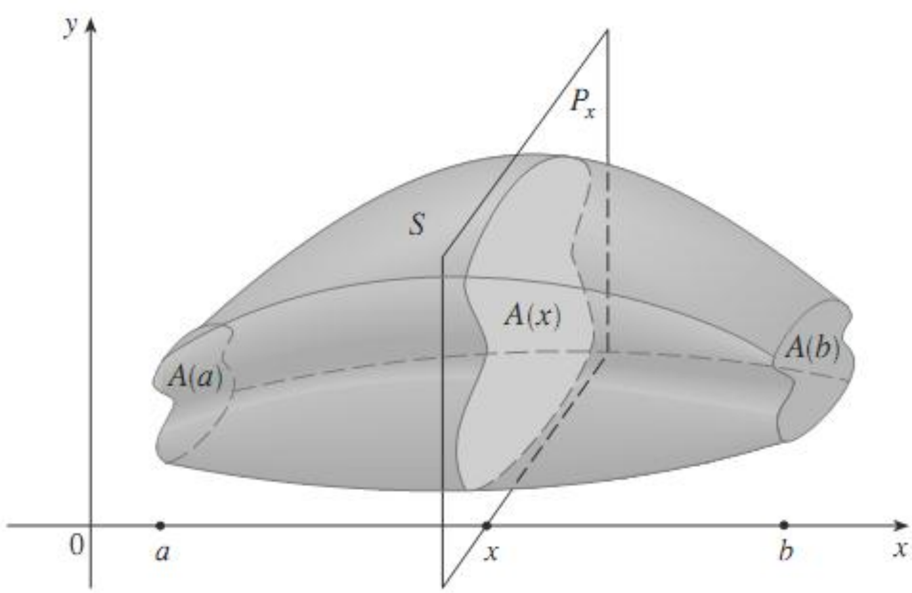




For the shapes above, the area of the face is the same at every point sliced parallel to the face. This is not always the case, and this is where calculus comes in.

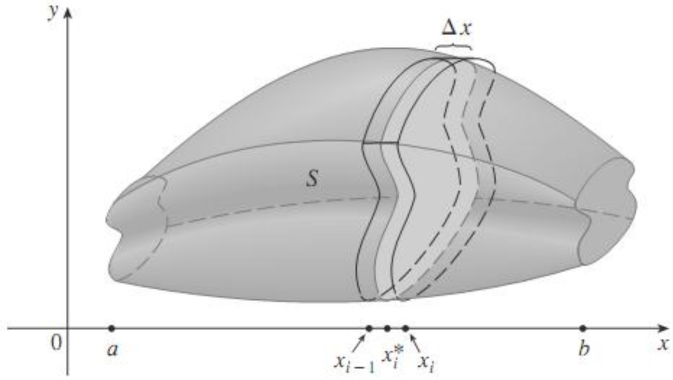
(Calculus enters stage left)

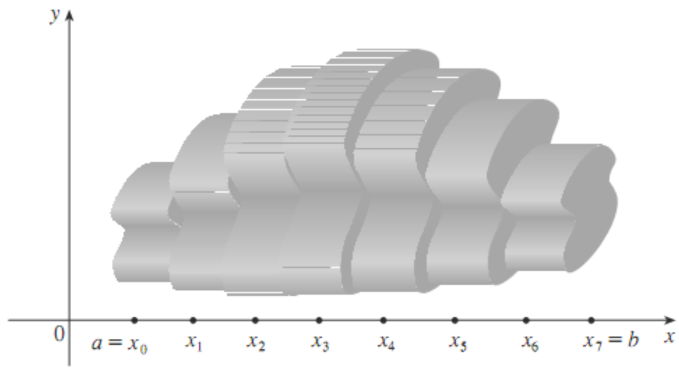
Imagine slicing a loaf of bread (mathematically). It might look like this



At each slice, we have a face with a different area, but one that might be a function of where the loaf was sliced along the horizontal.

For a finite slice, the measurable thickness of the slice, whether it be for a sandwich or Texas Toast, is Δx. This thickness forms the distance *h*, between the two faces. We can slice up the loaf from left to right using a uniform Δx.





We can approximate the volume of the entire loaf by finding the volumes of each slice $V(x_i) = A(x_i) \cdot \Delta x$ and adding them up (This works much better than finding volume by the Archimedean method of fluid displacement, which leaves the bread rather soggy. Incidentally, Archimedes is called the "Father of Integral Calculus" since he was the first person to envision finding volumes by this thin, slicing method).

As we slice the regions thinner and thinner and thinner, approaching infinitely thin, we lose the ability to sandwich a piece of meat between two sliced, but we also get increasingly better approximations of the volume. Here's the summary:

Definition of Volume

Let *S* be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of *S* in the plane through *x* and perpendicular to the *x*-axis is $A(x)$, where *A* is a continuous function, then the **volume** of *S* is

$$V = \lim_{\Delta x \to 0} \sum A(x_i)\Delta x = \int_a^b A(x)\,dx$$

Example 1:
The volume formulas for the shapes shown at the top of this lesson and the others from your geometry class (or related rate and optimization sections) are derived from calculus. Let's show that the formula for the volume a sphere of radius *r* is $V = \frac{4}{3}\pi r^3$.

Anytime our cross-sections, perpendicular to an **axis of rotation** (or **revolution**), are circles (or thin cylinders called **discs**), we can us a similar approach. Very often we will have to create/envision our solids by rotating or revolving a given region **around or about** an axis. ***When we create solids by revolving around an axis that is perpendicular to our slices, our cross-sections will always be circular.***

Disc Method for Volumes of Solids of Rotation

When the volume of solid is obtained by rotating a region **perpenDISCular** to the axis of rotation and the cross-sections are discs or circles, the volume of the solid is given by

$$V = \pi \int_a^b R(x)^2 \, dx$$

Where $R(x)$ is the radius of rotation as a function of *x*.

Example 2:

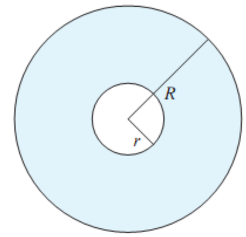
Find the volume of the solid formed by rotating the region bounded by the *x*-axis, $y = \sqrt{x}$, and $x = 1$ around the *x*-axis.

perpenDISCular

Example 3:
Find the volume of the solid formed by rotating the region bounded by the $y=1$, $y=\sqrt{x}$, and $x=0$ around the line $y=1$.

Example 4:
Find the volume of the solid obtained by rotating the region bounded by $y=x^3$, $y=8$, and $x=0$ about the *y*-axis.

Example 5:
What if we were to take the region from the previous example and rotate it around the *x*-axis instead of the *y*-axis? What would the shape look like? What would a perpendicular slice look like? Find the volume of the solid obtained by rotating the region bounded by $y=x^3$, $y=8$, and $x=0$ about the *x*-axis.



Sometimes, our cross sections are circles but have a void or hole in them. In this case, our circular cross-section, perpendicular to the axis of rotation, will resemble a **washer**, with an inner, smaller radius *r*, and a larger, outer radius *R*.

In this case, the area of the face of the cross section will be

$$A(x) = \pi R(x)^2 - \pi r(x)^2 = \pi\left(R(x)^2 - r(x)^2\right)$$

Washer Method for Volumes of Solids of Rotation/Revolution

When the volume of solid is obtained by rotating a region **perpenWASHular** to the axis of rotation and the cross-sections are washers, the volume of the solid is given by

$$V = \pi\int_a^b \left[R(x)^2 - r(x)^2\right]dx$$

Where $R(x)$ is the larger, outer radius of rotation and $r(x)$ is the smaller, inner radius rotation.

Example 6:
The region in the first quadrant enclosed by the *y*-axis and the graphs of $y = \cos x$ and $y = \sin x$ is revolved about the *x*-axis to form a solid. Find its volume.

Important things to consider when using the Washer method:

- Draw a picture, draw a picture, draw a picture, . . . You must identify the region 1st!
- Like the Disc method, the cross-sections (slices/representative rectangles) must be PERPENDICULAR to the axis of rotation/revolution
- Before writing an equation for *R* and *r*, draw them on your diagram. If you can draw them, you can write them.
- When writing an equation for *R* and *r*, it will still involve $TOP - BOTTOM$ (vertical slice) or $RIGHT - LEFT$ (horizontal slice). **One of these in each case will be the axis of rotation itself.**
- **DON'T FORGET TO SQUARE EACH RADIUS BEFORE SUBTRACTING THEM.** The most common error is to integrate as $V = \pi\int_a^b \left[\left(R(x) - r(x)\right)^2\right]dx$. This is **WRONG**. Keep telling yourself that you're subtracting two separate volumes: $\pi R^2 dx - \pi r^r dx$. The π and dx are simply factored out.

perpenWASHular

Example 7:

Find the volume of the solid formed when the *R* enclosed by the curves $y = x$ and $y = x^2$ is rotated about the following axes:

(a) the *x*-axis.

(b) the line $y = 2$

(c) the line $y = -5$

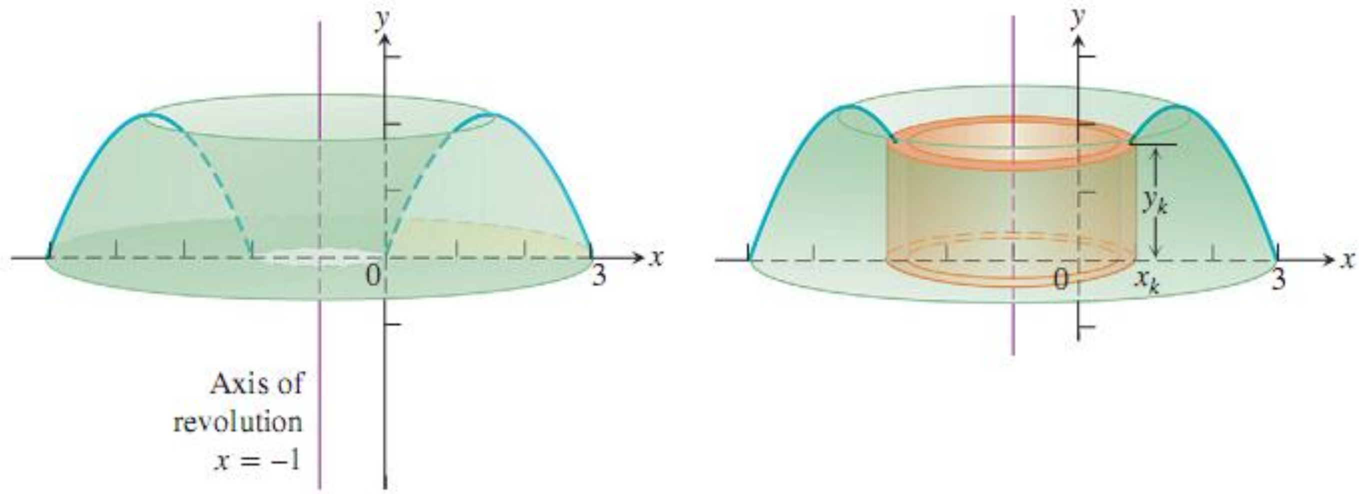
(d) the *y*-axis

(e) the line $x = -1$

(f) the line $x = 17$

Example 8:

The region enclosed by the *x*-axis and the parabola $f\left(x\right)=3x-x^2$ is revolved about the line $x=-1$ to generate a solid of revolution resembling a Bundt cake. HOW WOULD YOU PREFER TO SLICE THE REGION? What is the consequence of this choice? Is there a way to accommodate your slicing preference with the axis of rotation/revolution????? We MUST find the volume of this cake? (What would happen if the graph was rotated about the line $x=4$ instead?)



paraSHELL

Shell Method for Volumes of Solids of Rotation/Revolution

When the volume of solid is obtained by rotating a region **paraSHELL** to the axis of rotation and the cross-sections are cylindrical shells with radius $r(x)$ and height $h(x)$, the volume of the solid is given by

$$V = 2\pi \int_a^b r(x) \cdot h(x)\, dx$$

We don't need to worry about holes, since we are only integrating/slicing over the interval $[a,b]$, parallel to the axis of rotation.

For solids generated by revolving a region around an axis, we now have a method that will accommodate our slicing preference. For such problems, we choose our slicing method first, not the method. The method is determined by comparing your slicing preference to the axis of rotation:

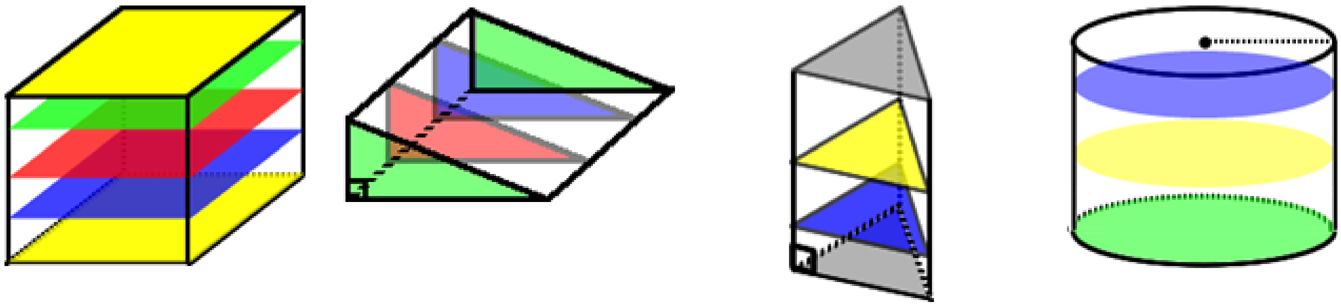
PerpenDISCular/PerpenWASHular or ParaSHELL??

Example 9:

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the *y*-axis.

So what if we don't generate our solid by revolving it around an axis? Remember these guys from the beginning of the lesson?

$$V = A \cdot h$$



If we can find the formula for the area of the cross-sectional face at any point along the infinitely thin slice, we can add them all up to find the volume. As a reminder:

Definition of Volume

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \int_a^b A(x)\,dx$$

Imagine that a concrete slab has been poured. Upon that slab, walls are built perpendicular to the slab. If we can find the area of the face of one of these walls, we kind find the volume of that panel, and thus, the entire house. The slab represents the area of the region enclosed by the curves.

If we can find $A(x)$ (for slices perpendicular to the x-axis) or $A(y)$ (for slices perpendicular to the y-axis), we just need to integrate over the interval $[a,b]$ with respect to x (dx being the infinitely thin width of each slice).

Example 10:

Find the volume of the solids whose bases are bounded by the graphs of $y = x + 1$ and $y = x^2 - 1$, with the following cross sections taken perpendicular to the *x*-axis. Identify the region that will be the base, find the points of intersection defining the region, then write an equation, $s(x) =$, for the **side** length of each cross section.

(a) Squares

(b) Rectangles, height of 2

(c) Rectangles, height is five times the base

(d) Quarter Circles

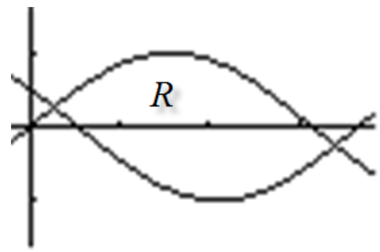
(e) Semicircles

(f) Isosceles Right Triangles, base is a short leg

(g) Isosceles Right Triangles, hypotenuse is the base

(h) Equilateral Triangles

Example 11:



(Calculator) An oil spill on the surface of the water has a surface shape***, R,*** defined by the intersections of the equations $f(x) = \sin x$ and $g(x) = -\sin\left(x - \frac{\pi}{6}\right)$ as shown in the figure. The depth of the oil spill at each value of *x*, measured perpendicular to the *x*-axis, has a depth given by $D(x) = 2\cos(x/2)$. Find the volume of the oil spill.

Example 12:

A region *R*, defined by the intersections of the graphs of $y = 5x$, $y = -\frac{x}{5} + 3$, and $y = 0$, is the base of the solid. For this solid, at each *y*, the cross section perpendicular to the *y*-axis has area $A(y) = y^2 + 1$. Find the volume of the solid.