

§6.6—Improper Integrals

We've studied definite integrals this year, and, as you recall, they were all pretty routine, easy, and very courteous. Now we study improper integrals. As the name implies, they will be less routine, less easy, and perhaps lift boxes the wrong way.

Let's start by getting your calculator warmed up.



Example 1:

Evaluate by using your calculator.

$$(a) \int_1^{100} \frac{1}{x} dx =$$

$$(b) \int_1^{1000} \frac{1}{x} dx =$$

$$(c) \int_1^{1,000,000} \frac{1}{x} dx =$$

Based on your values above, what do you think $\int_1^{\infty} \frac{1}{x} dx$ equals?

$$(d) \int_1^{100} e^{-x} dx =$$

$$(e) \int_1^{1000} e^{-x} dx =$$

What do you think $\int_1^{\infty} e^{-x} dx$ equals?

f) Sketch the graphs of $f(x) = \frac{1}{x}$ and $f(x) = e^{-x}$ in the first quadrants, and shade the regions represented

by $\int_1^{\infty} \frac{1}{x} dx$ and $\int_1^{\infty} e^{-x} dx$.

Why would these two integrals give such different results?

Now that that's got you irked, try the following

How to deal with infinite intervals of integration:

1. If $\int_a^b f(x)dx$ exists of every $b > a$, then $\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$, provided the limit exists as is finite.

2. If $\int_b^c f(x)dx$ exists of every $b < c$, then $\int_{-\infty}^c f(x)dx = \lim_{b \rightarrow -\infty} \int_b^c f(x)dx$, provided the limit exists as is finite.

Example 2:

$$(a) \int_1^\infty \frac{1}{x^{3/2}} dx =$$

$$(b) \int_1^\infty \frac{1}{x^2} dx =$$

$$(c) \int_1^\infty \frac{1}{x^3} dx =$$

Do you see a pattern as a function of the exponent??

Fact:

If $a > 0$, then $\int_a^\infty \frac{1}{x^p} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$. These are called p -series integrals.

If $a = 1$ and $p > 1$, then $\int_1^\infty \frac{1}{x^p} dx$ **converges** to $\frac{1}{p-1}$

Note: For those with a fancy for bathroom humor, it is interesting to note that the p -test relies so heavily on the number one. No elaboration needed.

Example 3:

$$(a) \int_1^\infty \frac{1}{x^{2/3}} dx =$$

$$(b) \int_1^\infty \frac{1}{x^{1.1}} dx =$$

$$(c) \int_1^\infty x^{-7} dx =$$

$$(d) \int_1^\infty 3 \cdot x^{-3/2} dx$$

Example 4:

$$\int_{-\infty}^0 \frac{1}{\sqrt{3-x}} dx$$

Integrals such as $\int_a^{\infty} f(x)dx$, $\int_{-\infty}^a f(x)dx$, and $\int_{-\infty}^{\infty} f(x)dx$ are called **improper integrals**, not because they lift boxes incorrectly or even curse like a sailor but because of one of three reasons:

1. They have an infinite interval of integration.
2. They have a discontinuity on the interior of the interval of integration
3. Both 1) and 2).

They are evaluated by rewriting the integral as a proper integral and then using limits. Not every improper integral equals a finite number. In fact, you'd probably expect anything integrated to or from infinity will be infinite. An improper integral that equals a finite value is said to **converge** to that value. An improper integral that does not equal a finite number is said to **diverge**.

Improper integrals with an infinite interval of integration are easy to spot.

Example 5:

$$(a) \int_1^{\infty} e^{-x} dx =$$

$$(b) \int_{-\infty}^0 e^{x/4} dx =$$

$$(c) \int_1^{\infty} \frac{1}{x} dx =$$

Example 6:

Let $f(x) = e^{-2x}$ for $0 \leq x < \infty$, and let R be the unbounded region in the first quadrant below the graph of f . Find the volume of the solid generated when R is revolved around the x -axis.

Example 7:

“Rapid” fire . . . Watch out for indeterminate forms!!!!

$$(a) \int_1^{\infty} (1-x)e^{-x} dx =$$

$$(b) \int_0^{\infty} e^{-x} \sin x dx =$$

$$(c) \int_0^{\infty} \frac{2dx}{x^2 + 4x + 3} =$$

Example 8:

Determine if the following converge or diverge. If they converge, find the value to which they converge.

This might be helpful:

Convergent plus Convergent = Convergent

Divergent plus Convergent = Divergent

Divergent plus Divergent = Divergent

($\infty + \infty$ or $-\infty - \infty$)

Divergent – Divergent = Indeterminate

($\infty - \infty$ or $-\infty + \infty$)

$$(a) \int_1^{\infty} \frac{2+x}{x^2} dx$$

$$(b) \int_1^{\infty} \left[\frac{1}{x} - \frac{3x}{1+5x^2} \right] dx$$

Sometimes, an integral can be doubly improper.

If $\int_{-\infty}^c f(x) dx$ and $\int_c^{\infty} f(x) dx$ are both convergent, then $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$, where c is any number. Symmetry can also be used to circumvent the doubleness of the impropriety. Note as well that this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

Example 9:

$$(a) \int_{-\infty}^{\infty} \frac{dx}{1+x^2} =$$

$$(b) \int_{-\infty}^{\infty} xe^{-x^2} dx =$$

Example 10:

(Calculator permitted) Find the area of the region bounded by the graph of $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and the x -axis.

Sometimes we cannot find the antiderivative of an integrand of an improper integral. We might be able to draw a conclusion if we can compare it to something similar for which we know something.

Comparison Test for Convergence or Divergence

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$, then

- If $\int_a^{\infty} g(x) dx$ converges, then $\int_a^{\infty} f(x) dx$ converges too!
- If $\int_a^{\infty} f(x) dx$ diverges, then $\int_a^{\infty} g(x) dx$ diverges too!

Example 11:

(a) If $0 \leq e^{-x^2} \leq e^{-x}$ for all $x \geq 1$, determine if $\int_1^{\infty} e^{-x^2} dx$ converges or diverges.

(b) Determine if $\int_{\pi}^{\infty} \frac{2 + \cos x}{x} dx$ converges or diverges.

(c) Determine if $\int_4^{\infty} \frac{2}{x + e^x} dx$ converges or diverges.

(d) Determine if $\int_4^{\infty} \frac{2}{x - e^{-x}} dx$ converges or diverges.

When an integral is improper has a finite interval of integration, it is improper because its interval spans an infinite discontinuity (vertical asymptote). These are harder to spot, so be vigilant!!



Example 12:

(a) Evaluate $\int_0^1 x^{-1/3} dx$ by evaluating the following on your calculator:

(i) $\int_{0.01}^1 x^{-1/3} dx$

(ii) $\int_{0.001}^1 x^{-1/3} dx$

(iii) $\int_{0.0001}^1 x^{-1/3} dx$

(b) Evaluate $\int_0^1 x^{-3} dx$ by evaluating the following on your calculator:

(i) $\int_{0.01}^1 x^{-3} dx$

(ii) $\int_{0.001}^1 x^{-3} dx$

(iii) $\int_{0.0001}^1 x^{-3} dx$

As you can see, if a function of the form $\frac{1}{x^p}$ tends to converge as $x \rightarrow \infty$ will tend to diverge as $x \rightarrow 0^+$ at the vertical asymptote. The exception, of course is $\frac{1}{x}$, which diverges in both directions.

When we recognize an infinite discontinuity at an endpoint, we have to set up a one-sided limit. When the infinite discontinuity is on the interior, we have to set up two integrals, approaching the VA from each side in each integral.

Example 13:

Verify your results from Example 12 by evaluating the following improper integrals.

(a) $\int_0^1 \frac{1}{x^{1/3}} dx =$

(b) $\int_0^1 \frac{1}{x^3} dx =$

Example 14:

(a)
$$\int_0^{27} \frac{dx}{\sqrt[3]{27-x}} =$$

(b)
$$\int_0^3 \frac{dx}{(x-1)^{2/3}} =$$

Example 15:

What is the area of the region in the fourth quadrant bounded by the graphs of $y = 2 \ln x$, $y = 0$, and $x = 0$?

Again, symmetry can be your friend . . .

Example 16:

Find the area bounded by the graph of $y = \frac{1}{x}$ and the x -axis over the interval $-2 \leq x \leq 2$.

Example 17:

Find the length of the curve defined by $y = \sqrt{4 - x^2}$ over the interval $0 \leq x \leq 2$. Use your result to find the circumference of the circle given by $x^2 + y^2 = 4$.

Sometimes an integral can be Dastardly Doubly Improper

Example 18:

Determine if the following integral is convergent or divergent. If it is convergent, find its value.

$$\int_0^{\infty} \frac{1}{x^2} dx$$

Example 19:

Let $f(x) = \frac{1}{x}$ for $1 \leq x < \infty$, and let R be the unbounded region in the first quadrant below the graph of f .

Find the volume of the solid generated when R is revolved around the x -axis. (Note: The region is known as Gabriel's Horn or **Torricelli's Trumpet**.)



*It's really unlike me to
toot my own horn.*