§7.1—Intro to Parametric & Vector Calculus

In Algebra, equations are graphed in two variables, x and y. Now we will graph equations with x, y, and t, or with x, y, and θ, where x and y are expressed independently in terms of t or θ. The third variable, t or θ is called the parameter, and the separate equation are called parametric equations.

Example 1:
Without a calculator, make a table, and sketch the curve, indicating the direction of your graph. Then eliminate the parameter. Verify on your calculator.

$$x = t^2 - 4 \text{ and } y = \frac{t}{2},\ -2 \le t \le 3$$

What do you notice about the graphs of $x = 4t^2 - 4$ and $y = t$, $-1 \le t \le 1.5$

What do you notice about the graphs of $x = 4(2\sin t + 1)^2 - 4$ and $y = 2\sin t + 1$, $-1.571 \le t \le 0.253$

While rectangular equations on restricted intervals show the ______________, parametric equations show

the __________________________, ________________________, and ____________________________.

Example 2:
Do the same for $x = \frac{1}{\sqrt{t+1}}$, $y = \frac{t}{t+1}$

Example 3:
Do the same for $x = 2 + 3\cos t$, $y = -1 + 2\sin t$

Time to B.O.T.C.!

Parametric Equations & Formulas for Calculus

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at the point (x, y) is given by $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ where $\frac{dx}{dt} \neq 0$, and the second derivative is given by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{dy}{dx}\right] \cdot \frac{dt}{dx} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}.$$

Example 4:

Without a calculator, given $x = 2\sqrt{t}$, $y = 3t^2 - 2t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and evaluate at $t = 1$.

Example 5:

Without a calculator, given $x = 4\cos t$, $y = 3\sin t$, write an equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$.

Example 6:

Without a calculator, find all points of horizontal and vertical tangency given $x = t^2 + t$, $y = t^2 - 3t + 5$.

Parametric Arc Length

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ is the length of the arc from $t = a$ to $t = b$

Example 7:

Without a calculator, find the arc length of the given curve if $x = t^2$, $y = 4t^3 - 1$, $0 \le t \le 1$.

It's time to revisit particle motion.

Horizontal and Vertical Velocity Component VECTORS

- $x'(t) = \dfrac{dx}{dt}$ is the rate at which the *x*-coordinate is changing with respect to *t* or the velocity of a particle in the horizontal direction.

- $y'(t) = \dfrac{dy}{dt}$ is the rate at which the *y*-coordinate is changing with respect to *t* or the velocity of a particle in the vertical direction.

- $\vec{s} = \langle x(t), y(t) \rangle = (x(t), y(t))$ is the position at any time *t*.

- $\vec{v} = \langle x'(t), y'(t) \rangle = (x'(t), y'(t))$ is the velocity vector at any time *t*.

- $\vec{a} = \langle x''(t), y''(t) \rangle = (x''(t), y''(t))$ is the acceleration vector at any time *t*.

  *note: the vectors may or may not be contained within the chevrons $\langle \ \rangle$.

- $\dfrac{dy}{dx}$ is the rate of change of *y* with respect to *x* or the slope of the tangent line to the curve or the slope of the position vector.

- $\dfrac{d^2y}{dx^2}$ is the rate of change of the slope of the curve with respect to *x*.

- $|\vec{v}(t)| = \|\vec{v}(t)\| = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ is the **<u>speed of a particle</u>** or the **<u>magnitude</u>** or the **<u>length</u>** or the **<u>norm</u>** of the velocity vector.

- $\int_a^b \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$ is the **length of the arc** from $t = a$ to $t = b$ or the **distance traveled** by a particle from $t = a$ to $t = b$.

  *Remember that $\int_a^b |v(t)|\, dt$ is the total distance traveled, whether it be along a straight line or curve.

Example 8:
(No Calculator) A particle moves in the $xy-$plane so that at any time *t*, $t \geq 0$, the position of the particle is given by $x(t) = t^3 + 4t^2$, $y(t) = t^4 - t^3$.

(a) Find the velocity vector at $t = 1$, (b) the speed of the particle at $t = 1$, and (c) the acceleration vector at $t = 1$.

Example 9:
(No Calculator) A particle moves in the $xy-$plane so that $x = \sqrt{3} - 4\cos t$ and $y = 1 - 2\sin t$, where $0 \leq t \leq 2\pi$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

(b) The path of the particle intersects the *x*-axis twice. Write an expression that represents the distance traveled by the particle between the two *x*-intercepts. Do not evaluate.