§9.1—Sequences & Series: Convergence & Divergence

A sequence is simply list of things generated by a rule

More formally, a **sequence** is a function whose domain is the set of positive integers, or **natural numbers**, *n*, such that $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The range of the function are called the terms in the sequence,

$$a_1, a_2, a_3, \ldots, a_{n, \ldots}$$

Where a_n is called the ***n*th term** (or rule of sequence), and we denote the sequence by $\{a_n\}$.

The sequence can be expressed by either
1) an ample number of terms in the sequence, separated by commas
2) an explicit function defined by the **rule of sequence**
3) the rule of sequence set off in braces.

Example 1:
The sequence $2, 4, 6, 8, \ldots$ is the sequence of even numbers. Express the same sequence as a rule of a non-negative integer *n*. The sequence $1, 3, 5, \ldots$ is the sequence of odd numbers. Express the same sequence as a rule of a non-negative integer *n*. How many in the list are needed to establish the "rule" in the absence of the explicitly-stated rule?

***NOTE: When given a sequence as a list, the first term is usually designated to be associated with $n = 1$. This is because we are using *n* as an ordinal (or counting) number, rather than a cardinal number.

We will be primarily interested in what happens to the sequence for increasingly large values of *n*.

Example 2:
If $a_n = \left\{\dfrac{4n}{3+2n}\right\}$, list out the first five terms, then estimate $\lim\limits_{n\to\infty} a_n$.

FACT:

Let $\{a_n\}$ be a sequence of real numbers.

Possibilities:

1) If $\lim\limits_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity
2) If $\lim\limits_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity
3) If $\lim\limits_{n\to\infty} a_n = c$, an finite real number, then $\{a_n\}$ converges to c
4) If $\lim\limits_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation

Definition:

$n!$ is read as "n factorial." It is defined recursively as $n! = n(n-1)!$ or as

$$n! = n(n-1)! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

Por ejemplo: $9! = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

Example 3:

Determine whether the following sequences converge or diverge.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$

(c) $a_n = 3 + (-1)^n$

(d) $a_n = \frac{n}{1-2n}$

(e) $a_n = \frac{\ln n}{n}$

(f) $a_n = \frac{n!}{(n+2)!}$

(g) $a_n = \frac{2n!}{(n-1)!}$

(h) $a_n = \frac{n + (-1)^n}{n}$

(i) $a_n = \frac{(-1)^n (n-1)}{n}$

(j) $a_n = \frac{2^n}{(n+1)!}$

(k) $a_n = \left(1 + \frac{1}{n}\right)^n$

(l) $\left\{\frac{(2n)!}{n^n}\right\}$

Sometimes, albeit rarely, we have to write the rule of sequence as a function of *n* from a pattern.

Example 4:
Write an expression for the *n*th term.

(a) 3, 8, 13, 18, . . .

(b) 5, -15, 45, -135, . . .

(c) 1, 4, 9, 16, 25, . . .

(d) 4, 10, 28, 82, . . .

(e) $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

(f) $\ln 1,\ \ln 2,\ \ln 4,\ \ln 8, \ldots$

A **Series** is the sum of the terms in a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely. A series is informally the result of adding any number of terms from a sequence together: $a_1 + a_2 + a_3 + \cdots$. A series can be written more succinctly by using the summation symbol sigma, $\sum$., the Greek letter "S" for **Esum** (the "E" is both silent and not really there.)

For infinite series, we can look at the sequence **of partial sums**, that is, looking to see what the sums are doing as we add additional terms. In general, the *n*th partial sum of a series is denoted S_n. This can be explored on a calculator by adding sequential terms to the aggregate sum.

Example 5:
For both $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$, generate the sequence of partial sums $S_1,\ S_2,\ S_3,\ \ldots,\ S_n$, for each, then determine if the sequences converges or diverges. Do the results surprise you? Where else have we seen something like this before?

Convergence and Divergence of a Series

What does it mean for a series to converge? To diverge? Let's look at a couple series from a special family called **geometric series**.

Example 6:

Given the series $\sum_{n=1}^{\infty}\frac{3}{2^n}=\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\frac{3}{128}+\frac{3}{256}+\frac{3}{512}+\frac{3}{1024}+\cdots$,

find the first **ten** terms of the sequence of partial sums, and list them below, $S_1, S_2, S_3, \ldots, S_{10}$. Based on this sequence of partial sums, do you think the series converges? Diverges? To what? (HINT: first rewrite the rule of sequence so that it looks like an **exponential function** of *n*.)

Example 7:

Given the series $\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)^n=\frac{3}{2}+\frac{9}{4}+\frac{27}{8}+\frac{81}{16}+\frac{243}{32}+\cdots$, find the first **<u>five</u>** terms of the sequence of partial sums, and list them below. Based on this sequence of partial sums, do you think the series converges? Diverges? To what?

We are now going to look at several families of infinite series and several tests that will help us determine whether they converge or diverge. For some that converge, we might be able to give the actual sum, or an interval in which we know the sum will be. For others, simply knowing that they converge will have to suffice.

Geometric Series, nth Term Test for Divergence, and Telescoping Series

Geometric Series Test (GST)

A geometric series is in the form $\sum_{n=0}^{\infty} a \cdot r^n$ or $\sum_{n=1}^{\infty} a \cdot r^{n-1}$, $a \neq 0$

The geometric series **diverge**s if $|r| \geq 1$.

If $|r| < 1$, the series **converge**s to the sum $S = \frac{a_1}{1-r}$.

Where a_1 is the first term, regardless of where *n* starts, and *r* is the common ration.

Example 8:

Using the GST, determine whether the following series converge or diverge. If the converge, find the sum.

(a) $\sum_{n=1}^{\infty} \frac{3}{2^n}$ (b) $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ (c) $\sum_{n=2}^{\infty} 3\left(-\frac{1}{2}\right)^n$

***n*th Term Test for Divergence (ONLY)**

If $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

(think about it, it should make perfect sense!)

Note: This does **NOT** say that if $\lim_{n \to \infty} a_n = 0$, then the series **DOES** converge. This test can only be used to prove that a series diverges (hence the name.) If $\lim_{n \to \infty} a_n = 0$, then this test doesn't tell us anything, is inconclusive, doesn't work, fails, etc. . . . We MUST use another test. This test can be a GREAT time-saver. Always perform it **FIRST**, not second, but FIRST!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

Example 9:
Determine whether the following series converge or diverge. If they converge, find their sum.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{2n+3}{3n-5}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{2n!+1}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n-2}{3^n}$ (d) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{(1.1)^n}$

A series such as $\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$ is called a **telescoping series** because it collapses to one term or just a few terms. If a series collapses to a finite sum, then it converges by the **Telescoping Series Test**.

Example 10:
Determine whether the following series converges or diverges. If they converge, find their sum.

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3}$

Integral Test and *p*-Series

Integral Test

If f is **D**ecreasing, **C**ontinuous, and **P**ositive (**D**ogs **C**uss in **P**rison!) for $x \geq 1$ AND $a_n = f(x)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either BOTH converge or diverge.

Note 1: This does NOT mean that the series converges to the value of the definite integral!!!!!!
Note 2: The function need only be decreasing for all $x > k$ for some $k \geq 1$.

If the series converges to S, then the remainder, $R_n = |S - S_n|$ is bounded by $0 \leq R_n \leq \int_n^{\infty} f(x)dx$. (Not on AP exam, but on my exam.). This means $S \in [S_n, S_n + R_n]$.

Example 11:

Determine whether the following series converge or diverge. If they converge, find an interval in which the sum resides using S_4.

(a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Example 12:

Approximate the sum of the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ by using six terms. Include an estimate of the maximum error for your approximation.

***p*-series**

A series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$ is called a *p*-series, where *p* is a positive constant.

For $p=1$, the series $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ is called the **harmonic series**.

Based on your experience with *p*-series and their reliance on the number one, fill in chart below.

***p*-Series Test**

The *p*-series $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$, based on above

a) If $p=1$, ______________________________

b) If $p<1$, ______________________________

c) If $p>1$, ______________________________

Note: If the *p*-series converges, we cannot find its sum ☹ . This is more often the case than not.

Example 13:

Determine of the following converges or diverges:

(a) $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ (b) $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n}}$ (c) $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n}$ (d) $\sum_{n=1}^{\infty}\frac{999999999}{n^{1.0000000001}}$

Based on your experience with improper integrals, again, fill in the chart below.

Comparison of Series

Direct Comparison Test (DCT) If $a_n \geq 0$ and $b_n \geq 0$, 1) If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ ______________________. 2) If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ ______________________. **NOTE:You must state/show the inequality when stating the conclusion of the test!!**

Example 14:

Determine whether the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

(c) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

(d) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$

(e) $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

(f) $\sum_{n=2}^{\infty} \frac{1}{n^4-10}$

Sometimes the inequalities needed above don't hold or are difficult to show, but you still strongly suspect the result because you recognize a similar series with which to compare it.

Limit Comparison Test (LCT)

If $a_n \geq 0$ and $b_n \geq 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ **or** $\lim_{n\to\infty} \frac{b_n}{a_n} = L$, where L is both finite and positive.

Then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

Example 15:

Determine whether the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$

(b) $\sum_{n=1}^{\infty} \frac{n^4 + 10}{4n^5 - n^3 + 7}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n^3 - 2}$

(d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$

Alternating Series

An alternating series is a series whose terms are alternately positive and negative on **consecutive** terms.

For instance: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ and $-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!}$

In general, just knowing that $\lim\limits_{n\to\infty} a_n = 0$ tells us very little about the convergence of the series $\sum\limits_{n=1}^{\infty} a_n$; however, it turns out that an alternating series must converge if its terms consistently shrink in size and approach zero!!

Alternating Series Test (AST)

If $a_n > 0$, then the alternating series $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ or $\sum\limits_{n=1}^{\infty}(-1)^{n+1} a_n$ converges if both of the following conditions are satisfies:

1) $\lim\limits_{n\to\infty} a_n = 0$

2) $\{a_n\}$ is a decreasing (or Non-increasing) sequence; that is, $a_{n+1} \le a_n$ for all $n > k$, for some $k \in \mathbb{Z}$

Note: This does NOT say that if $\lim\limits_{n\to\infty} a_n \ne 0$ the series DIVERGES by the AST. The AST can ONLY be used to prove convergence. If $\lim\limits_{n\to\infty} a_n \ne 0$, then the series diverges, but by the ***n*th-term** test NOT the AST.

Example 16:

Determine whether the following series converge or diverge.

(a) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1} n}{2n-1}$ (b) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n} n}{\ln(2n)}$ (c) $\sum\limits_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

(d) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$ (e) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n}}{(n-5)^2+1}$ (f) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

ABOSOLUTE VS CONDITIONAL CONVERGENCE

Crazy Fact: Sometimes a mere rearrangement of terms in a convergent alternating series can yield different sums!!!

Theorem:

If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

Such a series is called **absolutely convergent**. Notice that if it converges on its "own," the alternator only allows it to converge more "rapidly".

$\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example 17:
Determine whether the given alternating series converges or diverges. If it converges, determine whether it is absolutely convergent or conditionally convergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n}$

Alternate Series Remainder

Suppose an alternating series satisfies the conditions of the AST, namely that $\lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is not increasing. If the series has a sum *S*, then $|R_n| = |S - S_n| \le a_{n+1}$, where S_n is the *n*th partial sum of the series.

In other words, if an alternating series satisfies the conditions of the AST, you can approximate the sum of the series by using the *n*th partial sum, S_n, and your error will have an absolute value no greater than the first term left off, a_{n+1}. This means $S \in [S_n - R_n, S_n + R_n]$

Example 18:

Approximate the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ by using its first six terms, and find the error. Use your results to find an interval in which *S* must lie.

Example 19:

Approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with an error less than 0.001

Ratio and Root Tests

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

1. $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
2. $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$
3. The ratio test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Series involving expressions that grow very rapidly such as factorials and/or exponential work especially well with the Ratio Test.

Example 20:

Determine if the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ (b) $\sum_{n=1}^{\infty} \frac{n^2 3^{n+1}}{2^n}$ (c) $\sum_{n=1}^{\infty} \frac{(n+1)!}{3^n}$

<u>Root Test</u>

1. $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$
2. $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$
3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

If the entire rule of sequence can be written as power of n, the Root Test is hard to beat!

Example 21:

(a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

(b) $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n}\right)^n$

Example 22:
Putting it all together. Determine if the following series converge or diverge. Name the test used and the criteria of each test used.

(a) $\sum_{n=1}^{\infty} \frac{1+3n^2+n^3}{4n^3-5n+2}$

(b) $\sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n$

(c) $\sum_{n=1}^{\infty} \frac{4}{n^3}$

(d) $\sum_{n=1}^{\infty} \frac{n^2}{5^n}$

(e) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5+5}}$

(f) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$

(g) $\sum_{n=1}^{\infty} \frac{5n^2-6n+3}{n^3-7n+8}$

(h) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$

(i) $\sum_{n=1}^{\infty} \frac{3^n+4}{2^n}$

(j) $\sum_{n=1}^{\infty} \frac{8n^3-6n^5}{12n^4-9n^5}$

(k) $\sum_{n=1}^{\infty} \sqrt{\frac{3n+1}{n^5+2}}$

(l) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n2^n}$

(m) $\sum_{n=1}^{\infty} \left(\frac{2n}{5n-1}\right)^n$

(n) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

(o) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$

(p) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$

To help you remember all these tests, just think of Moses fleeing with the Israelites from Pharaoh:

PARTING C

P p-series: Is the series in the form $\frac{1}{n^p}$?

A Alternating series: Does the series alternate? If it does, are the terms getting smaller, and is the nth term 0?

R Ratio Test: Does the series contain things that grow very large as n increases (exponentials or factorials)?

T Telescoping series: Will all but a couple of the terms in the series cancel out?

I Integral Test: Can you easily integrate the expression that defines the series (are Dogs Cussing in Prison?)

N nth Term divergence Test: Is the nth term something other than zero?

G Geometric series: Is the series of the form $\sum_{n=0}^{\infty} ar^n$?

C Comparison Tests: Is the series ***almost*** another kind of series (e.g. p-series or geometric)? Which would be better to use: the Direct or Limit Comparison Test?