§9.2—Taylor Polynomials

Taylor Polynomials and Approximations

Polynomial functions can be used to approximate other elementary functions such as $\sin x$, e^x, and $\ln x$.

Example 1:

Find the equation of the tangent line for $f(x)=\sin x$ at $x=0$, then use it to approximate $\sin(0.2)$. Is this an over or an under approximation of $\sin(0.2)$?

The equation of the tangent line used in Ex. 1 is called a **first-degree Taylor polynomial**. Taylor polynomials of higher degree can be used to obtain increasingly better approximations of non-polynomial functions within a certain **radius** from a **center of approximation** $x=c$.

Example 2:

On your calculator graph $y1=\sin x$. Use the following window: $X[-9,9]$, $Y[-4,4]$. Now in $y2=$, graph successively, adding an extra term each time, the following: $y2: x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$

What do you notice? What is $y1(0)$? $y2(0)$? What is $y1(0.2)$? $y2(0.2)$?

<u>Definition of an *n*th-degree Taylor polynomial:</u>

If *f* has *n* derivatives at $x = c$, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the <u>*n*th-degree Taylor polynomial for *f* centered at *c*</u>, named after Brook Taylor, an English mathematician.

Note 1: A first-degree Taylor polynomial is a tangent line to *f* at *c*.

Note 2: $\frac{f^{(n)}(c)}{n!}$ **is the coefficient of the** $(x-c)^n$ **term**

If $c = 0$, then $P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$ is called the <u>*n*th-degree Maclaurin polynomial for *f*</u>, named after Scottish mathematician, Colin Maclaurin.

Note 3: Maclaurin not only got his name on a very specific case of Taylor's work, but he also had big hair.
Note 4: For Taylor & Mac Polynomials, you MUST use a "squiggle." For example, $f(x) \approx P_n(x) = \cdots$

(from hardycalculus.com)
It seems only natural to write about these two prolific mathematicians in the same Historical Sketch because they are both best known for their work on representations of functions by infinite series. These representations, known as Taylor series and Maclaurin series, are part of the standard fare of every calculus course.

Brook Taylor was born in 1685 into wealthy, cultured, but strict English family of Edmonton, Middlesex, England, now a part of Greater London, and was educated by private tutors up to the time he enrolled in St. John's College of Cambridge University. At Cambridge, he studied law but also became engaged with mathematics, and he graduated with a law degree in 1709. But by this time, he had already written a mathematics research paper and was becoming known in the mathematical world. Indeed, in **<u>1712</u>** he not only stated his method for his infinite series, be in this year he was both elected to the Royal Society and appointed to a committee to adjudicate the dispute between Leibniz and Newton concerning calculus priority. In 1714, he was elected Secretary to the Royal Society and held that position for four years, a period that was his most productive in terms of mathematical output.

Taylor is given credit for conceiving the concept of the calculus of finite differences, the tool of integration by parts, and of course the Taylor series representation of functions. He suffered from problems of health, and he died on December 29, **<u>1731</u>** at the age of 46.

Colin Maclaurin was born in 1698 into a minister's family residing in the village of Kilmodan, in Argyllshire, a county of western Scotland. He was a child prodigy who at the age of eleven became a student at the University of Glasgow. Then at the age of nineteen, he was elected Professor of Mathematics at Marischal College in Aberdeen, and two years after that, he was elected to the Royal Society of London and came to know Isaac Newton. In fact, it was Newton who recommended to the University of Edinburgh that Maclaurin be made a professor of mathematics there, and he was, in 1725. In 1740, he shared a French Academy of Sciences prize with Leonhard Euler and Daniel Bernoulli. In **<u>1742</u>**, Maclaurin wrote a two-volume treatise defending Newton's mathematics as a rebuttal to Bishop George Berkeley, who had claimed that Newton used faulty reasoning. That is the book in which the special case of Taylor series appears, now called Maclaurin series.

When in 1745 organized supporters of King James II marched on Edinburgh, Maclaurin engaged himself in defending the city, after which he escaped to England. This effort destroyed his health, and he died on June 14, 1746 at age 48.

Example 3:
Find the Maclaurin polynomial of degree $n = 5$ for $f(x) = \sin x$. Then use $P_5(x)$ to approximate the value of $\sin(0.1)$ using correct notation. Find the error for your approximation and determine an interval in which $\sin(0.1)$ could actually live. Finally, compare your approximation to the actual value of $\sin(0.1)$. Is it in your interval? Cool, huh?!

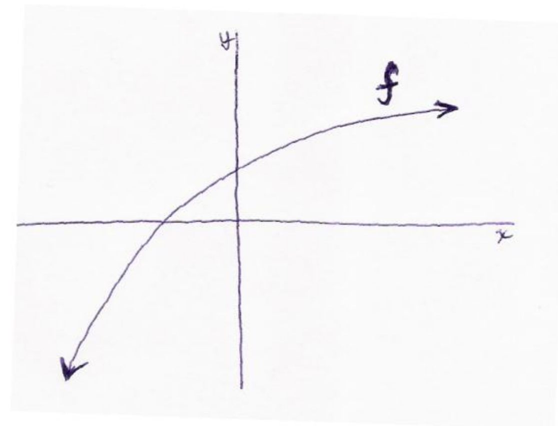
Example 4:
Find the Taylor polynomial of degree $n = 6$ for $f(x) = \ln x$ at $c = 1$. Then use $P_6(x)$ to approximate the value of $\ln(1.1)$

Example 5:
Suppose that g is a function which has continuous derivatives, and that $g(2)=3$, $g'(2)=-4$, $g''(2)=7$, $g'''(2)=-5$. Write the Taylor polynomial of degree 3 for g centered at 2.

Example 6:
Use a third-degree Taylor approximation of e^x for x near 0 to find $\lim_{x\to 0}\frac{e^x-1}{2x}$, then compare it to the actual limit at zero.. Is this surprising? Why or why not.

Example 7:

Given that $P_2(x) = a + bx + cx^2$ is the second-degree Taylor polynomial for *f* about $x = 0$, what can you say about the signs of *a*, *b*, and *c* if *f* has the graph pictured at the right? Justify your answer.



Sometimes we can create Maclaurin Polynomials to approximate functions without having to derive them using Taylor's Theorem, but rather by modifying existing polynomials.

Example 8:

List the first four non-zero terms of the Maclaurin Polynomials for $f(x) = \sin x$, $f(x) = \cos x$, and $f(x) = e^x$, the find the following Maclaurin Polynomials.

(a) $g(x) = \sin(2x)$ (b) $g(x) = x\cos(x)$ (c) $g(x) = 4e^{x^2}$