§9.4—Power Series II: Geometric Series

A particularly important skill to develop for the AP exam, other than checking that you're in RADIAN mode, is to represent certain types of rational functions as a **geometric series**. Rather than producing the power series using Taylor's Rule, you will want to develop the series by manipulating a geometric series or, in some cases, using Long Division.

Example 1:
First we'll do a quick review of geometric series. Geometric series are formed by multiplying by a common ratio *r*.

(a) Suppose I told you to start with $a_1 = 1$ and to let $r = \frac{2}{3}$, what geometric series would you write? What would the sum be?

(b) What if $a_1 = 1$ and $r = -\frac{2}{3}$?

(c) What if $a_1 = 1$ and $r = x$?

Example 2:
Verify your answer from Example 1(c) by finding the power series for $\frac{1}{1-x}$ centered at $c = 0$ by

(a) using Taylor's Rule
(b) using LOOOONG DIVISION.
(c) Find the radius and interval of convergence. Verify by graphing.

Example 3:

Find a power series for $\dfrac{1}{1+x}$ centered at $c=0$, then find the interval of convergence. Include the first four nonzero terms and the general term.

Example 4:

Find a power series that represents $\dfrac{x}{1+x}$ centered at $c=0$, then find the interval of convergence. Include the first four nonzero terms and the general term.

Example 5:

Find a power series for $f(x)=\dfrac{1}{1-x^2}$ centered at $c=0$, then find the interval of convergence. Find the first four nonzero terms and the general term.

When you replace *x* with a multiple of *x*, beware a change in the radius and interval of convergence. . .

Example 6:

Find a power series that represents $\frac{1}{1-2x}$ centered at $c=0$, then find the interval of convergence. Include the first four nonzero terms and the general term.

Example 7:

Find a power series for $g(x)=\frac{1}{4+x}$ centered at $c=0$, then find the interval of convergence. Include the first four nonzero terms and the general term.

Sometimes we cannot center our function at $x=0$. In this case, we must try to rewrite our function with the new center showing.

Example 8:

Find a power series that represents $\frac{1}{x}$ centered at $c=1$, then find the interval of convergence. Include the first four nonzero terms and the general term.

Example 9:

(A bit of a booger) Find a power series for $h(x)=\dfrac{15}{2x-1}$, centered at $c=1$, then find the interval of convergence. Include the first four nonzero terms and the general term.

We can integrate or differentiate a power series to obtain a new series. When we do this, the radius of convergence will be the same, but the interval may change. In this case, we must retest the endpoints.

Example 10:

Find a power series that represents $\dfrac{1}{(1-x)^2}$ centered at $c=0$. Hint: what is $\dfrac{d}{dx}\left[\dfrac{1}{1-x}\right]$? What is the radius of convergence? Interval of convergence?

Example 11:

Find a power series that represents $\ln(1-x)$ centered at $c=0$. Hint: what is $\int\left(\dfrac{1}{1-x}\right)dx$? What is the radius of convergence? Interval of convergence?

Example 12:

(Similar to 2008—BC6B) Let f be the function given by $f(x)=\dfrac{1}{1+x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x=0$.

(b) Does the series found in part (a), when evaluated at $x=1$, converge to $f(1)$? Explain why or why not.

(c) The derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$. Write the first four nonzero terms of the Taylor series for $\arctan x$ about $x=0$.

(d) Use the series found in part (c) to find a rational number A such that $\left|A-\arctan\left(\dfrac{3}{4}\right)\right|<\dfrac{1}{10}$. Justify your answer.

Besides finding the sum of an infinite, convergent geometric series (and the telescoping series), there is one last way we are expected to find such sums: by recognizing a given infinite, convergent series as a Taylor series evaluated at a particular value of *x*.

Example 13:
By recognizing each series as a Taylor series evaluated at a particular value of *x*, find the sum of each of the following infinite, convergent series.

(a) $1-\frac{\pi^2}{2!}+\frac{\pi^4}{4!}-\frac{\pi^6}{6!}+\cdots$

(b) $1+\ln 4+\frac{\ln^2 4}{2!}+\frac{\ln^3 4}{3!}+\frac{\ln^4 4}{4!}+\cdots$

(c) $2\left(\frac{\pi}{6}\right)-\frac{2}{3!}\left(\frac{\pi}{6}\right)^3+\frac{2}{5!}\left(\frac{\pi}{6}\right)^5-\frac{2}{7!}\left(\frac{\pi}{6}\right)^7+\cdots$