§9.5—Lagrange Error Bound

Lagrange Form of the Remainder (also called Lagrange Error Bound or Taylor's Theorem Remainder)

Suppose we didn't have a calculator, but we were interested in the value of $\sin 1$. How could we approximate its value?

- Eyeball it on the graph?
- Compare it to $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}=???$
- Tangent line approximation?
- Euler's method?
- Taylor polynomial of degree >1?

Suppose we choose the third or fifth option, after all, polynomials are relatively easy to evaluate. Is there a way, depending on the degree of the polynomial we decide to use, to determine how accurate our approximation is? That is, do we know what **remains** from our approximation to get the actual value?

YES! Yes, we do . . . kind of.

When a Taylor polynomial, $T_n\left(x\right)$, centered at $x=c$ is used to approximate a function, $f\left(x\right)$, at a value $x=a$ near the center, we can express our result as follows:

If:

Function = Polynomial Approximation + Remainder,

Then:

Remainder = Function – Polynomial Approximation

Written mathematically, if:

$$f(a)=T_n(a)+R_n(a)$$

Then:

$$R_n(a)=f(a)-P_n(a)$$

We define the remainder, $R_n(a)$, to be:

$$R_n(a)=\left|T_n\left(a\right)-f\left(a\right)\right|=\left|\frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}(a-c)^{n+1}\right|$$

Taylor's Theorem

If a function f is differentiable through order $n+1$ in an interval containing the center $x=c$, then for each $x=a$ in the interval, there exists a number $x=z$ between a and c such that

$$f(a)=f(c)+f'(c)(a-c)+\frac{f''(c)}{2!}(a-c)^2+\cdots+\frac{f^{(n)}(c)}{n!}(a-c)^n+R_n(a)$$

where the remainder $R_n(a)$ is given by $R_n(a)=\left|\frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}(a-c)^{n+1}\right|$, called the **Lagrange Remainder (or Lagrange Error Bound).**

Historically, the remainder was not due to Taylor but to a French mathematician, **Joseph Louis Lagrange** (1736-1813), the MVT guy.

Historical Note: Up to the time of Mr. Lagrange, there seemed to have been very little interest in determining the error between exact and approximate values, but rather, there was a shared enthusiasm of "good feeling" for the rapidity of approximations and the goodness of the values they were finding.

Langrange wanted to quantify this "good feeling," and first published his estimation of the remainder for the Taylor series in his ***Théorie des fonctions analytiques*** in 1797. Remember that Brook Taylor introduced his formula for generating polynomial terms in 1712!

Langrange's efforts, thus ended the 85 year Era of "Good Feelings," (not to be confused with the period in American History that coincided with Monroe's presidency.)

For this reason, $R_n(a)$ is called the **Lagrange form** of the remainder.

a hmm, hmm, hmm, hmm.

When applying Taylor's Formula, we would not expect to be able to find the exact value of z. Rather, we are merely interested in a safe upper bound (maximum value) for $\left|f^{(n+1)}(z)\right|$ from which we will be able to tell how large the remainder $R_n(a)$ is.

Example 1:

(Calculator Permitted) Let f be a function with 5 derivatives on the interval $[2,3]$. Assume that $\left|f^{(5)}(x)\right| < 0.3$ for all x in the interval $[2,3]$ and that a fourth-degree Taylor polynomial, $T_4(x)$, for f at $c = 2$ is used to estimate $f(3)$.

(a) How accurate is this approximation? Give four decimal places.

(b) Suppose that $T_4(3) = 1.763$. Use your answer from (a) to find an interval in which $f(3)$ must reside.

(c) Could $f(3)$ equal 1.768? Why or why not?

(d) Could $f(3)$ equal 1.761? Why or why not?

Sometimes the value of $\left|f^{(n+1)}(z)\right|$ on the interval between our center, *c*, and the *x*-value at which we would like to evaluate our function, *a*, is, itself, not explicitly given nor easy to find. **In this case, we may choose the first known, reasonable rational number greater than** $\left|f^{(n+1)}(z)\right|$. If we choose a number that is too large, we defeat the purpose of the error bound, but if we choose a value smaller than $\left|f^{(n+1)}(z)\right|$, there is no guarantee that our approximation will fall in our interval.

Example 2:
(Calculator Permitted) Let's return to our original quandary—approximating $\sin 1$ with an idea of how accurate our approximation would be.

(a) Find the fifth-degree Maclaurin polynomial for $\sin x$. Then use your polynomial to approximate $\sin 1$, and use Taylor's Theorem to find the maximum error for your approximation. Give five decimal places.

(b) Use your answer from (a) to find an interval $\left[a,b\right]$ such that $a \le \sin 1 \le b$.

(c) Could $\sin 1$ equal 0.87? Why or why not?

Example 3:
(No Calculator)
(a) Write the fourth-degree Maclaurin polynomial for $f(x)=e^x$. Then use your polynomial to approximate $\sqrt{e}$, and find a Lagrange error bound for the maximum error when $|x|\le 0.5$.

(b) Use your answer from (a) to find an interval $[a,b]$ such that $a\le\sqrt{e}\le b$.

Example 4:
(Calculator Permitted) The function *f* has derivatives of all orders for all real numbers *x*. Assume that $f(2)=6$, $f'(2)=4$, $f''(2)=-7$, $f'''(2)=8$.

(a) Write the third-degree Taylor polynomial for *f* about $x=2$, and use it to approximate $f(2.3)$. Give three decimal places.

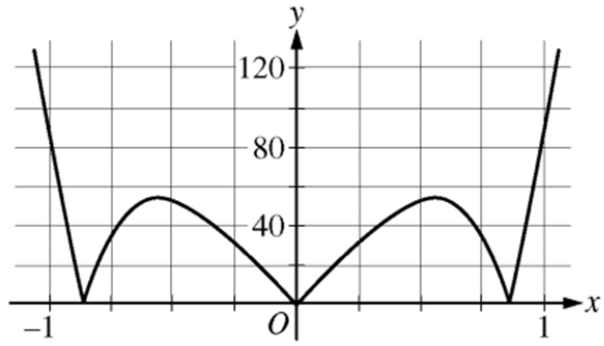
(b) The fourth derivative of *f* satisfies the inequality $\left|f^{(4)}(x)\right|\le 9$ for all *x* in the closed interval $[2,2.3]$. Use the Lagrange error bound on the approximation of $f(2.3)$ found in part (a) to find an interval $[a,b]$ such that $a\le f(2.3)\le b$. Give three decimal places.

(c) Based on the information above, could $f(2.3)$ equal 6.992? Explain why or why not.

Example 5:

(2011 BC6) Let $f(x)=\sin\left(x^2\right)+\cos x$.

The graph of $y=\left|f^{(5)}(x)\right|$ is shown at right.



Graph of $y=\left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x=0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x=0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x=0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x=0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x=0$. Using information from the graph of $y=\left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right)-f\left(\frac{1}{4}\right)\right|<\frac{1}{3000}$.