§10.1—Trigonometric Substitution

We have seen integrals similar to $\int x\sqrt{4-x^2}\,dx$. For such an integral, we can integrate quickly by recognizing the pattern ("off" by a -2), or we can do a formal *u*-substitution, which would replace the old "complex" inside function with a single variable. In this case, we would let $u = 4 - x^2$.

Often, though, integrals such as $\int \sqrt{4-x^2}\,dx$ show up (for instance, when finding the area of a circle or ellipse). This is a much more difficult integral than the first type. For such an integral, we will use a process called **inverse substitution**.

Rather than replacing a complex-looking function with a single variable, we will replace a single variable with a more complex-looking function—making it look more complex in order to make it easier to work with. How will that work? In this case, it will involve **trigonometric substitution**.

The goal of trig sub (for short) is to get rid of the radical in the integrand by way of the Pythagorean Identities, creating a single squared term rather than two terms. Recall that

$$\boxed{\begin{aligned} 1-\sin^2\theta &= \cos^2\theta \\ 1+\tan^2\theta &= \sec^2\theta \\ \sec^2\theta - 1 &= \tan^2\theta \end{aligned}}$$

The left side in each of the above identities resembles the form of each radicand in the integrand, with a proper trig sub, we will be able to transform the radicand into something resembling the right side. Let's explore.

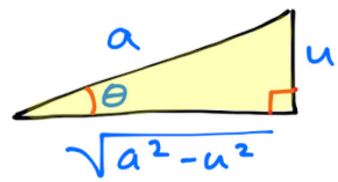
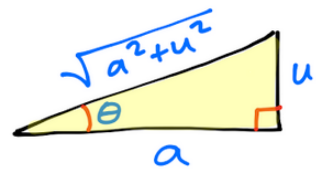
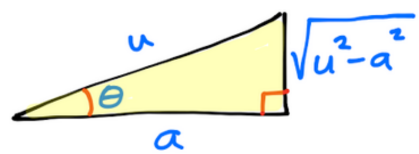
Example 1:

Given $\sqrt{4-x^2}$, determine which Pythagorean Identity form from above the radicand resembles, then determine a proper substitution to transform the radicand into a single squared term. Based on your trig substitution, draw a reference triangle and label all three sides in terms of *x*.

In general, for the type of radical expression of the form $\sqrt{a^2-u^2}$, where *a* is any constant and *u* is any function of *x*, we can make the substitution $u = a\sin\theta$ to obtain the following

$$\sqrt{a^2-u^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2\left(1-\sin^2\theta\right)} = \sqrt{a^2\cos^2\theta} = a\left|\cos\theta\right|$$

Trig Subs

Expression	Substitution	Identity Used	Triangle Drawn
$\sqrt{a^2-u^2}$	$u = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$	
$\sqrt{a^2+u^2}$	$u = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$	
$\sqrt{u^2-a^2}$	$u = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$	

Note: When we make these substitutions with functions of θ, we are doing so over the principal value ranges of arcsine, arctangent, and arcsecant. This will assure the substitution is a one-to-one function.

Example 2:

Evaluate $\int\sqrt{4-x^2}\,dx$. You might need some of the Trig Identities at right.

$$\cos^2\theta = \frac{1}{2}\left(1+\cos 2\theta\right)$$

$$\sin^2\theta = \frac{1}{2}\left(1-\cos 2\theta\right)$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Example 3:

Evaluate $\int \frac{\sqrt{16-x^2}}{x^2}\,dx$

Example 4:

Evaluate $\int \frac{dx}{\sqrt{9x^2+1}}$

Example 5:

Evaluate $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$

There needn't be a radical to use trig sub . . .

Example 6:

Evaluate $\displaystyle\int \frac{2}{\left(x^2+1\right)^2}\,dx$