

## §10.1—Trigonometric Substitution

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We have seen integrals similar to  $\int x\sqrt{4-x^2} dx$ . For such an integral, we can integrate quickly by recognizing the pattern (“off” by a  $-2$ ), or we can do a formal  $u$ -substitution, which would replace the old “complex” inside function with a single variable. In this case, we would let  $u = 4 - x^2$ .

Often, though, integrals such as  $\int \sqrt{4-x^2} dx$  show up (for instance, when finding the area of a circle or ellipse). This is a much more difficult integral than the first type. For such an integral, we will use a process called **inverse substitution**.

Rather than replacing a complex-looking function with a single variable, we will replace a single variable with a more complex-looking function—making it look more complex in order to make it easier to work with. How will that work? In this case, it will involve **trigonometric substitution**.

The goal of trig sub (for short) is to get rid of the radical in the integrand by way of the Pythagorean Identities, creating a single squared term rather than two terms. Recall that

$1 - \sin^2 \theta = \cos^2 \theta$ $1 + \tan^2 \theta = \sec^2 \theta$ $\sec^2 \theta - 1 = \tan^2 \theta$
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The left side in each of the above identities resembles the form of each radicand in the integrand, with a proper trig sub, we will be able to transform the radicand into something resembling the right side. Let’s explore.

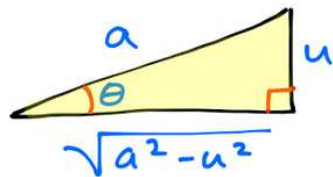
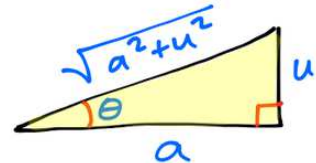
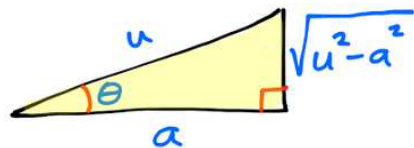
### Example 1:

Given  $\sqrt{4-x^2}$ , determine which Pythagorean Identity form from above the radicand resembles, then determine a proper substitution to transform the radicand into a single squared term. Based on your trig substitution, draw a reference triangle and label all three sides in terms of  $x$ .

In general, for the type of radical expression of the form  $\sqrt{a^2 - u^2}$ , where  $a$  is any constant and  $u$  is any function of  $x$ , we can make the substitution  $u = a \sin \theta$  to obtain the following

$$\sqrt{a^2 - u^2} = \sqrt{a^2 - a^2 \sin^2 \theta} = \sqrt{a^2(1 - \sin^2 \theta)} = \sqrt{a^2 \cos^2 \theta} = a|\cos \theta|$$

**Trig Subs**

Expression	Substitution	Identity Used	Triangle Drawn
$\sqrt{a^2 - u^2}$	$u = a \sin \theta$	$1 - \sin^2 \theta = \cos^2 \theta$	
$\sqrt{a^2 + u^2}$	$u = a \tan \theta$	$1 + \tan^2 \theta = \sec^2 \theta$	
$\sqrt{u^2 - a^2}$	$u = a \sec \theta$	$\sec^2 \theta - 1 = \tan^2 \theta$	

Note: When we make these substitutions with functions of  $\theta$ , we are doing so over the principal value ranges of arcsine, arctangent, and arcsecant. This will assure the substitution is a one-to-one function.

**Example 2:**

Evaluate  $\int \sqrt{4 - x^2} dx$ . You might need some of the Trig Identities at right.

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

**Example 3:**

Evaluate  $\int \frac{\sqrt{16-x^2}}{x^2} dx$

**Example 4:**

Evaluate  $\int \frac{dx}{\sqrt{9x^2+1}}$

**Example 5:**

Evaluate  $\int_{\sqrt{3}}^2 \frac{\sqrt{x^2 - 3}}{x} dx$

There needn't be a radical to use trig sub . . .

**Example 6:**

Evaluate  $\int \frac{2}{(x^2 + 1)^2} dx$