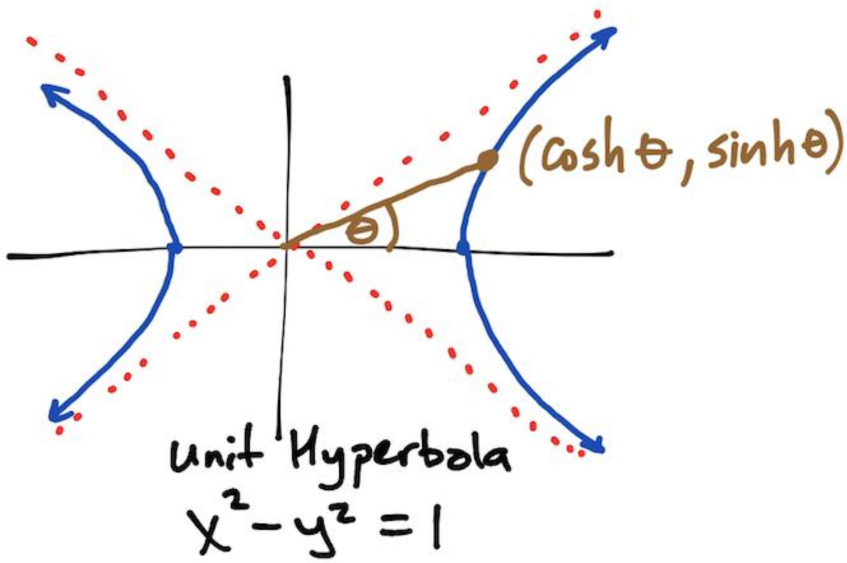
§10.6—Hyperbolic Functions

The circle has its trig functions, and the hyperbola has, what are known as, hyperbolic functions. On the Unit Circle, any point along the circle has the coordinate $(\cos q, \sin q)$. On a branch of the Unit Hyperbola, any point has the coordinate $(\cosh q, \sinh q)$.



Guess what the "h" is for . . .

We read $\cosh q$ as "hyperbolic cosine of theta," and $\sinh q$ is similarly read "hyperbolic sine of theta." Just as the circular trig functions show up in many real-world applications, so do the hyperbolic trig functions. In fact, many applications of exponential functions are really hyperbolic trig functions in disguise.

Because we will be talking about the hyperbolic **functions**, we will use *x* as the input, rather than q.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}$$

Notice that the functions $f(x)=\sinh x$ and $f(x)=\cosh x$ are the differences and the sums, respectively, of the two exponential functions $y=\frac{1}{2}e^{x}$ and $y=\frac{1}{2}e^{-x}$. Because of this, the graphs of $f(x)=\sinh x$ and $f(x)=\cosh x$ can be obtained by subtracting and adding the ordinates of the two exponential graphs.

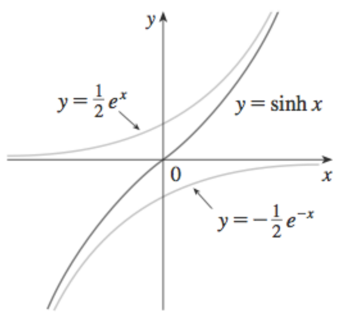


FIGURE 1
$y=\sinh x=\frac{1}{2}e^{x}-\frac{1}{2}e^{-x}$

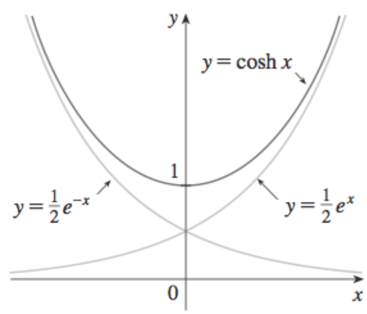


FIGURE 2
$y=\cosh x=\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}$

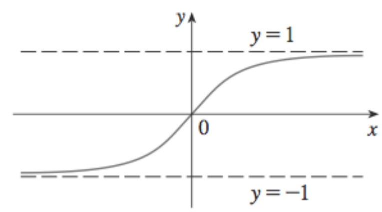
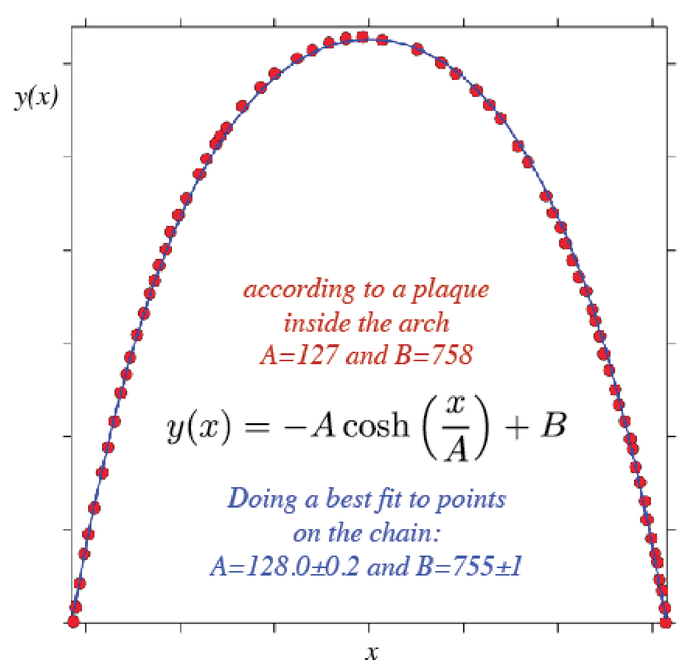


FIGURE 3
$y=\tanh x$

Example 1:
Find the domain and range and any symmetry for the three hyperbolic functions shown above.

Notice how the graph of $y=\cosh x$ resembles a parabola. This mistaken identity is quite easy to make, especially without quantitative analysis. The graph of $y=\cosh x$ is actually called a **catenary curve**, from the Latin *catena*, meaning "chain." This is because a heavy chain (or cable) suspended between two fixed points at the same elevation will take the sagging shape of a catenary with equation $y=a\cosh\left(\frac{x}{a}\right)$.

The most famous catenary (and mistaken parabola) of them all is the St. Louis/Gateway Arch.



Example 2:
Using the definition of $y = \cosh x$ and $y = \sinh x$, simplify $\cosh^2 x - \sinh^2 x$.

Just as there are many circular trig identities (and proofs), so there are many hyperbolic trig identities. For a list of many more, click here.

Let's talk calculus:

Example 3:
Using the definitions, find the derivatives of $y = \sinh x$ and $y = \cosh x$.

Example 4:
Using the definition, find the derivative of $y = \tanh x$.

Here are the derivatives of the Hyperbolic Functions

$$\frac{d}{dx}\left[\sinh u\right]=\left(\cosh u\right)u' \qquad \frac{d}{dx}\left[\coth u\right]=-\left(\operatorname{csch}^2 u\right)u'$$

$$\frac{d}{dx}\left[\cosh u\right]=\left(\sinh u\right)u' \qquad \frac{d}{dx}\left[\operatorname{sech} u\right]=-\left(\operatorname{sech} u\tanh u\right)u'$$

$$\frac{d}{dx}\left[\tanh u\right]=\left(\operatorname{sech}^2 u\right)u' \qquad \frac{d}{dx}\left[\operatorname{csch} u\right]=-\left(\operatorname{csch} u\coth u\right)u'$$

Example 5:

(a) $\frac{d}{dx}\left[\sinh\left(x^2-3\right)\right]=$ (b) $\frac{d}{dx}\left[\ln\left(\cosh x\right)\right]=$ (c) $\frac{d}{dx}\left[x\sinh x-\cosh x\right]=$

Example 6:

Evaluate $\int \cosh 2x\sinh^2 2x\,dx=$