Name Key Date________ Period______

Worksheet 2.10—Derivatives of Log Functions & LOG DIFF

Show all work. No calculator unless otherwise stated.

Short Answer

1. Find the derivative of each function with respect to x, given that a is a constant

(a) $y = x^a$

$y' = ax^{a-1}$

(b) $y = a^x$

$y = a^x \cdot \ln a$

$y = \ln a \cdot a^x$

(c) $y = x^x$

$\ln y = x \ln x$

$\frac{d}{dx}: \frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}(1)$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln x + 1$

$\frac{dy}{dx} = (\ln x + 1) \cdot y$

$\frac{dy}{dx} = x^x(\ln x + 1)$

(d) $y = a^a$

$y' = 0$

2. Evaluate each of the following. Remember to simplify early and often (especially when you have logs).

(a) $\frac{d}{dx}\left[e^{2\ln x}\right] =$

$\frac{d}{dx}\left[\left(e^{\ln x}\right)^2\right]$

$\frac{d}{dx}\left[x^2\right]$

$2x$

(b) $\frac{d}{dx}\left[\log_a a^{\sin x}\right] =$

$\frac{d}{dx}\left[\sin x\right]$

$\cos x$

(c) $\frac{d}{dx}\left[\log_2 8^{x-5}\right] =$

$\frac{d}{dx}\left[(x-5)\log_2 2^3\right]$

$\frac{d}{dx}\left[3(x-5)\right]$

$\frac{d}{dx}\left[3x-15\right]$

3

3. For each of the following, find $\frac{dy}{dx}$. Look to simplify using the properties of logs first.

(a) $y = \log_3 \frac{x\sqrt{x-1}}{2}$

$y = \log_3\left[\frac{1}{2} x \cdot (x-1)^{1/2}\right]$

$y = \log_3 \frac{1}{2} + \log_3 x + \frac{1}{2}\log_3(x-1)$

$y = \log_3 \frac{1}{2} + \frac{1}{\ln 3} \cdot \ln x + \frac{1}{2\ln 3} \cdot \ln(x-1)$

$\frac{dy}{dx} = \frac{1}{\ln 3} \cdot \frac{1}{x} + \frac{1}{2\ln 3} \cdot \frac{1}{x-1}$

$\frac{dy}{dx} = \frac{1}{x\ln 3} + \frac{1}{\ln 8(x-1)}$

(b) $y = x^{3/2}\log_2\sqrt{x+1}$

$y = x^{\frac{3}{2}} \cdot \frac{1}{2}\log_2(x+1)$

$y = \frac{1}{2}x^{\frac{3}{2}}\log_2(x+1)$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{3}{2}\right)x^{\frac{1}{2}} \cdot \log_2(x+1) + \frac{1}{2}x^{\frac{3}{2}} \cdot \frac{1}{\ln 2(x+1)}$

$\frac{dy}{dx} = \frac{3}{4}x^{\frac{1}{2}} \cdot \log_2(x+1) + \frac{x^{\frac{3}{2}}}{2\ln 2(x+1)}$

(c) $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

$y = \ln \cos x - \ln(\cos x - 1)$

$\dfrac{dy}{dx} = \dfrac{1}{\cos x}(-\sin x) - \dfrac{1}{\cos x - 1}\cdot(-\sin x)$

$\dfrac{dy}{dx} = -\tan x + \dfrac{\sin x}{\cos x - 1}$

(d) $y = \ln\left(\ln\dfrac{1}{x}\right)$

$y = \ln(\ln 1 - \ln x)$

$y = \ln(-\ln x)$

$\dfrac{dy}{dx} = \dfrac{1}{-\ln x}\left(-\dfrac{1}{x}\right)$

$\dfrac{dy}{dx} = \dfrac{1}{x\ln x}$

(e) $y = \ln^3 x$

$y = (\ln x)^3$

$\dfrac{dy}{dx} = 3(\ln x)^2 \cdot \dfrac{1}{x}$

$\dfrac{dy}{dx} = \dfrac{3(\ln x)^2}{x}$

or

$\dfrac{dy}{dx} = \dfrac{3\ln^2 x}{x}$

(f) $y = x\ln x^2$

$y = 2x\ln x$

$\dfrac{dy}{dx} = 2\ln x + 2x\left(\dfrac{1}{x}\right)$

$\dfrac{dy}{dx} = 2\ln x + 2$

(g) $y = \log_3(1 + x\ln x)$

$\dfrac{dy}{dx} = \dfrac{1}{\ln 3(1 + x\ln x)}\left(1\cdot\ln x + x\left(\dfrac{1}{x}\right)\right)$

$\dfrac{dy}{dx} = \dfrac{\ln x + 1}{\ln 3(1 + x\ln x)}$

(h) $y = \ln\sqrt[4]{\dfrac{4x-2}{3x+1}}$

$y = \dfrac{1}{4}\ln(4x-2) - \dfrac{1}{4}\ln(3x+1)$

$\dfrac{dy}{dx} = \dfrac{1}{4(4x-2)}\cdot(4) - \dfrac{1}{4(3x+1)}(3)$

$\dfrac{dy}{dx} = \dfrac{1}{4x-2} - \dfrac{3}{4(3x+1)}$

4. Use implicit differentiation to find $\frac{dy}{dx}$.

(a) $x^2 - 3\ln y + y^2 = 10$

$\frac{d}{dx}: 2x - 3\frac{1}{y}\cdot\frac{dy}{dx} + 2y\cdot\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(2y - \frac{3}{y}\right) = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y - \frac{3}{y}}\left(\frac{y}{y}\right)$

$\frac{dy}{dx} = \frac{-2xy}{2y^2 - 3}$

or

$\frac{dy}{dx} = \frac{2xy}{3 - 2y^2}$

(b) $\ln xy + 5x = 30$

$\ln x + \ln y + 5x = 30$

$\frac{d}{dx}: \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$

$\frac{1}{y}\frac{dy}{dx} = -5 - \frac{1}{x}$

$\frac{dy}{dx} = y\left(-5 - \frac{1}{x}\right)$

$\frac{dy}{dx} = -5y\frac{x}{x} - \frac{y}{x}$

$\frac{dy}{dx} = \frac{-5xy - y}{x}$

or

$\frac{dy}{dx} = -\frac{5xy + y}{x}$

5. Find an equation of the tangent line to the graph of $x + y - 1 = \ln\left(x^2 + y\sqrt{2}\right)$ at $(1,0)$.

point: $(1,0)$

slope: $m = \frac{1}{1-\sqrt{2}}$

$x + y - 1 = \ln(x^2 + y\sqrt{2})$

$x + y - 1 = \ln(x^2 + \sqrt{2}y)$

$\frac{d}{dx}: 1 + \frac{dy}{dx} = \frac{1}{x^2 + \sqrt{2}y}\left(2x + \sqrt{2}\frac{dy}{dx}\right)$

Note: Easier if plug in $(1,0)$ now then solve for $\frac{dy}{dx}$)

$\left.\frac{d}{dx}\right|_{(1,0)}: 1 + \frac{dy}{dx} = 1\left(2 + \sqrt{2}\frac{dy}{dx}\right)$

$\frac{dy}{dx} - \sqrt{2}\frac{dy}{dx} = 1$

$\frac{dy}{dx}(1 - \sqrt{2}) = 1$

$\frac{dy}{dx} = \frac{1}{1-\sqrt{2}}$

Eq. of Tang. Line: $y = 0 + \frac{1}{1-\sqrt{2}}(x - 1)$

$1 + \frac{dy}{dx} = \left(x^{-2} + (\sqrt{2}y)^{-1}\right)\left(2x + \sqrt{2}\frac{dy}{dx}\right)$

$1 + \frac{dy}{dx} = 2x^{-1} + \sqrt{2}x^{-2}\frac{dy}{dx} + 2^{1/2}xy^{-1} + y^{-1}\frac{dy}{dx}$

6. A line with slope m passes through the origin and is tangent to $y = \ln\left(\frac{x}{3}\right)$. What is the value of m?

m is the derivative of $y = \ln\left(\frac{x}{3}\right)$

$\frac{dy}{dx} = \frac{1}{\left(\frac{x}{3}\right)}\left(\frac{1}{3}\right)$

$\frac{dy}{dx} = \frac{3}{x}\cdot\frac{1}{3}$

$\frac{dy}{dx} = \frac{1}{x}$

$m = \frac{y - 0}{x - 0}$

$\frac{1}{x} = \frac{y}{x}$

$y = 1$

$1 = \ln\frac{x}{3}$

$e = \frac{x}{3}$

$3e = x$

$m = \frac{1}{3e}$

7. Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

point: $(0,0)$

slope: $m =$

$y = e^x$

$\frac{dy}{dx} = e^x$

At $(0,0)$ $\frac{dy}{dx} = 1$

Eq. Tang. Line: $y = 0 + 1(x - 0)$

$(y = x)$

8. Find the point where the tangent line to the curve $y = e^{-x}$ is perpendicular to the line $-2x + y = 8$.

Tangent L: point:

slope: $m_T = -\frac{1}{2}$

$-2x + y = 8$

$y = 2x + 8$

$m = 2$

$m_T = -\frac{1}{2}$

$y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$

$-\frac{1}{2} = -e^{-x}$

$\frac{1}{2} = e^{-x}$

$\ln\frac{1}{2} = -x$

$-\ln\frac{1}{2} = x$

or

$\ln 2 = x$

$y = e^{-\ln 2} = \frac{1}{e^{\ln 2}} = \frac{1}{2}$

Tang. Line: $y = \frac{1}{2} - \frac{1}{2}(x - \ln 2)$

$2(2x + 8) = \left(\frac{1}{2} - \frac{1}{2}(x - \ln 2)\right)2$

$4x + 16 = 1 - x + \ln 2$

$5x = -15 + \ln 2$

$x = \frac{\ln 2 - 15}{5}$

$y = 2\left(\frac{\ln 2 - 15}{5}\right) + 8$

pt perp. @ $\left(\frac{\ln 2 - 15}{5}, 2\left(\frac{\ln 2 - 15}{5}\right) + 8\right)$

9. Use Logarithmic Differentiation to evaluate the following.

(a) $\frac{d}{dx}\left[\sqrt[5]{\frac{(x-3)^4(x^2+1)}{(2x-5)^3}}\right] =$

Let $y = \sqrt[5]{\frac{(x-3)^4 \cdot (x^2+1)}{(2x-5)^3}}$

$y = (x-3)^{4/5} \cdot (x^2+1)^{1/5} \cdot (2x-5)^{-3/5}$

$\ln y = \frac{4}{5}\ln(x-3) + \frac{1}{5}\ln(x^2+1) - \frac{3}{5}\ln(2x-5)$

$\frac{d}{dx}: \frac{1}{y} \cdot \frac{dy}{dx} = \frac{4}{5}\left(\frac{1}{x-3}\right) + \frac{1}{5}\left(\frac{1}{x^2+1}\right)(2x) - \frac{3}{5}\left(\frac{1}{2x-5}\right)(2)$

$\frac{dy}{dx} = y\left[\frac{4}{5(x-3)} + \frac{2x}{5(x^2+1)} - \frac{6}{5(2x-5)}\right]$

(b) If $y = x^{1/\ln x}$, find $\frac{dy}{dx}$.

$y = x^{\frac{1}{\ln x}}$

$\ln y = \frac{1}{\ln x} \cdot \ln x$

$\ln y = 1$

$\frac{d}{dx}: \frac{1}{y} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = 0$

10. Let $f(x) = \ln\left(1 - x^2\right)$.

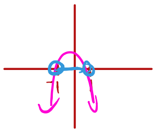
(a) State the domain of *f*.

$1 - x^2 > 0$

$1 - x^2 = 0$

$(1+x)(1-x) = 0$

$x = -1,\ x = 1$



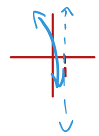
$D_f : (-1, 1)$

(b) Find $\lim\limits_{x \to -1^-} f(x)$

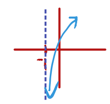
$\lim\limits_{x \to -1^-} \ln(1-x^2)$ or $\ln\left[\lim\limits_{x \to -1^-} (1-x^2)\right]$

$\ln(0)$?

$\lim\limits_{x \to -1^-} \ln[(1-x)\cdot(1+x)]$



$\lim\limits_{x \to -1^-} \ln(1-x) + \lim\limits_{x \to -1^-} \ln(1+x)$ DNE



$-\infty$

(c) Find $f'(x)$.

$f'(x) = \frac{1}{1-x^2}(-2x)$

$f'(x) = -\frac{2x}{1-x^2}$

(d) State the domain of $f'(x)$.

from c: $f'(x) = -\frac{2x}{1-x^2}$

Domain of f' can either be the same as f(x) or a SUBSET of f(x). It cannot be larger!

so $1 - x^2 \neq 0$ (i.e: include values not in D_f.)

on $(-\infty,-1) \cup (-1,1) \cup (1,\infty)$

already excluded from f so also excluded from f'!

$D_{f'} : (-1, 1)$

(d) Explain why $f''(x) < 0$ for all *x* in the domain of *f*.

Domain of f: $(-1,1)$

from c, $f'(x) = \frac{-2x}{1-x^2}$

$f''(x) = \frac{(1-x^2)(-2) - (-2x)(-2x)}{(1-x^2)^2}$

$f''(x) = -2\left(\frac{(1-x^2) + 2x^2}{(1-x^2)^2}\right)$

$f''(x) = -2\left(\frac{1+x^2}{(1-x^2)^2}\right)$

$> 0 \ \forall x \in D_f$.

Multiple Choice Thus $f''(x) = -2\left(\frac{1+x^2}{(1-x^2)^2}\right) < 0,\ \forall x \in D_f$.

__B__ 11. Use the properties of logs to simplify, as much as possible, the expression:

$$\log_a 32 + \frac{4}{5}\log_a 4 - \frac{4}{5}\log_a 2 + \log_a \frac{1}{2^{\frac{14}{5}}}$$

(A) $\log_a 128$ (B) $\log_a 8$ (C) $\log_a 32$ (D) $\log_a 2^{-7}$ (E) 8

$\log_a 32 + \log_a 4^{4/5} - \log_a 2^{4/5} + \log_a 2^{-\frac{14}{5}}$

$\log_a\left(\frac{32 \cdot 4^{4/5}}{2^{4/5}} \cdot 2^{-14/5}\right)$

$\log_a\left(\frac{2^5 \cdot 2^{8/5}}{2^{4/5} \cdot 2^{14/5}}\right)$

$\log_a 2^3$

$\log_a 8$

C 12. Simplify the expression: $2^{5(\log_2 e)\ln x}$

(A) 5^x (B) e^{11} (C) x^5 (D) x^{10} (E) x^2

$2^{5(\log_2 e)\ln x}$

$\left(2^{\log_2 e}\right)^{5\ln x}$

$e^{5\ln x}$

$\left(e^{\ln x}\right)^5$

x^5

Domain of f' is either the same as f or a subset. Never bigger!

D 13. Which of the following is the domain of $f'(x)$ if $f(x) = \log_2(x+3)$?

(A) $x < -3$ (B) $x \le 3$ (C) $x \ne -3$ (D) $x > -3$ (E) $x \ge -3$

$D_f: (-3, \infty)$ $x+3>0$ $x>-3$

$f'(x) = \frac{1}{x+3}(\ln 2)$

$f'(x) = \frac{\ln 2}{x+3}$

$x \ne -3$

$\cancel{(-\infty, -3)} \cup (-3, \infty)$ $D_f = D_{f'}$

already excluded!

Mess raised to a mess... USE LD!

A 14. If $f(x) = \left(x^2+1\right)^{(2-3x)}$, then $f'(1) =$

(A) $-\frac{1}{2}\ln(8e)$ (B) $-\ln(8e)$ (C) $-\frac{3}{2}\ln 2$ (D) $-\frac{1}{2}$ (E) $\frac{1}{8}$

$\ln f(x) = (2-3x)\ln(x^2+1)$

$\frac{d}{dx}: \frac{1}{f(x)} \cdot f'(x) = -3\ln(x^2+1) + (2-3x)\left(\frac{1}{x^2+1}\right)(2x)$

$\left.\frac{d}{dx}\right|_{x=1} \frac{1}{f(1)} \cdot f'(1) = -3\ln 2 + -1\left(\frac{1}{2}\right)(2)$ $f(1) = 2^{-1}$

$2 \cdot f'(1) = \ln \frac{1}{8} - 1$ $\ln e = 1$

$f'(1) = \frac{1}{2}\left(\ln\frac{1}{8} - \ln e\right)$

$f'(1) = \frac{1}{2}\ln\left(\frac{1}{8e}\right)$

$f'(1) = -\frac{1}{2}\ln(8e)$

D.S : $\infty - \infty$ IND

B 15. Determine if $\lim_{x\to\infty}\left[\ln(2+5x)-\ln(2+3x)\right]$ exists, and if it does, find its value.

(A) $\ln\frac{1}{2}$

(B) $\ln\frac{5}{3}$

(C) $\ln\frac{3}{5}$

(D) $\ln 2$

(E) Does Not Exist

$\lim_{x\to\infty} \ln\left(\frac{2+5x}{2+3x}\right)$

$\ln\left[\lim_{x\to\infty}\frac{5x+2}{3x+2}\right]$

$\ln\left(\frac{5}{3}\right)$

E 16. Find the derivative of $f(t)=\dfrac{2\ln t}{3+\ln t}$.

(A) $f'(t)=\dfrac{2}{t(3+\ln t)^2}$

(B) $f'(t)=\dfrac{6\ln t}{(3+\ln t)^2}$

(C) $f'(t)=\dfrac{6}{(3+\ln t)^2}$

(D) $f'(t)=\dfrac{2}{t(3+\ln t)}$

(E) $f'(t)=\dfrac{6}{t(3+\ln t)^2}$

$f'(t)=\dfrac{(3+\ln t)\left(\frac{2}{t}\right)-2\ln t\left(\frac{1}{t}\right)}{(3+\ln t)^2}$

$f'(t)=\dfrac{\frac{2}{t}\left(3+\ln t-\ln t\right)}{(3+\ln t)^2}$

$f'(t)=\dfrac{6}{t(3+\ln t)^2}$

mess to a mess... use LD!

B 17. Determine the derivative of *f* when $f(x)=x^{4x}$

(A) $f'(x)=(\ln x+4)x^{4x}$

(B) $f'(x)=4(\ln x+1)x^{4x}$

(C) $f'(x)=4(\ln x+1)$

(D) $f'(x)=(\ln x+1)x^{4x}$

(E) $f'(x)=4x^{4(x-1)}$

$\ln f(x)=4x\ln x$

$\frac{d}{dx}: \frac{1}{f(x)}\cdot f'(x)=4\ln x+4x\left(\frac{1}{x}\right)$

$\frac{1}{x^{4x}}\cdot f'(x)=4\ln x+4$

$f'(x)=4x^{4x}(\ln x+1)$

C 18. Find the derivative of *f* when $f(x)=x\left[7\sin(\ln x)+2\cos(\ln x)\right]$.

(A) $f'(x)=x\left[5\sin(\ln x)+9\cos(\ln x)\right]$

(B) $f'(x)=5\sin(\ln x)-9\cos(\ln x)$

(C) $f'(x)=5\sin(\ln x)+9\cos(\ln x)$

(D) $f'(x)=9\sin(\ln x)+5\cos(\ln x)$

(E) $f'(x)=x\left[9\sin(\ln x)+5\cos(\ln x)\right]$

$f(x)=x\left[7\sin(\ln x)+2\cos(\ln x)\right]$

$f'(x)=1\left[7\sin(\ln x)+2\cos(\ln x)\right]+\cancel{x}\left[7\cos(\ln x)\left(\cancel{\tfrac{1}{x}}\right)-2\sin(\ln x)\cdot\left(\cancel{\tfrac{1}{x}}\right)\right]$

$f'(x)=5\sin(\ln x)+9\cos(\ln x)$