WS P.3—Simplifying Expressions and Algebraic Gymnastics

Show all work on notebook paper. No Calculator

1. Find the exact value of each expression
   (a) \( \log_{10} 25 + \log_{10} 4 \)
   \[
   \log_{10} 100 \quad \log_{10} 16 \quad \frac{2}{2}
   \]
   (b) \( e^{4 \ln 2} \)
   \[
   e^{16} \quad e^{16} \quad 16 \quad 16
   \]

2. Solve each of the following equations for \( x \). Find the simplified, exact value.
   (a) \( e^x = 3 \)
   \[
   \frac{\ln(e^x)}{\ln(3)} = \ln(3) \quad x = \ln(3)
   \]
   (b) \( e^{e^x} = 3 \)
   \[
   \frac{\ln(e^{e^x})}{\ln(3)} = \ln(3) \quad e^x = \ln(3) \quad x = \ln(\ln(3))
   \]
   (c) \( \log_3 (x+1) = 2 \)
   \[
   3^{\log_3 (x+1)} = 3^2 \quad x+1 = 9 \quad x = 8
   \]
   (d) \( \log_3 27 = x \)
   \[
   x = \log_3 3^3 \quad x = 3
   \]

Multiple Choice

3. Rationalize the numerator of \( \frac{\sqrt{x+4} - \sqrt{x-2}}{x} \)
   (A) \( \frac{2}{x(\sqrt{x+4} + \sqrt{x-2})} \)
   (B) \( \frac{6}{x(\sqrt{x+4} - \sqrt{x-2})} \)
   (C) \( \frac{6}{x(\sqrt{x+4} + \sqrt{x-2})} \)
   (D) \( \frac{2x}{\sqrt{x+4} + \sqrt{x-2}} \)
   (E) \( \frac{6x}{\sqrt{x+4} - \sqrt{x-2}} \)

\[
\frac{\sqrt{x+4} - \sqrt{x-2}}{x} \times \frac{\sqrt{x+4} + \sqrt{x-2}}{\sqrt{x+4} + \sqrt{x-2}}
\]

\[
\frac{(x+4) - (x-2)}{x(\sqrt{x+4} + \sqrt{x-2})}
\]

\[
\frac{x+4 - x+2}{x(\sqrt{x+4} + \sqrt{x-2})}
\]

\[
\frac{6}{x(\sqrt{x+4} + \sqrt{x-2})}
\]
4. Which, if any, of the following statements are true when \(a, b\) are real numbers?

I. For all positive \(a\) and \(b\), \(\sqrt{a+b} = \sqrt{a} + \sqrt{b}\).

II. For all \(a\) and \(b\), \((a+b)^2 = |a+b|\).

III. For all positive \(a\) and \(b\), \(\frac{a-b}{\sqrt{a} + \sqrt{b}} = \sqrt{a} - \sqrt{b}\).

(A) III only     (B) all of them     (C) I and II only     (D) I only     (E) II and III only

(F) none of them     (G) I and III only     (H) I only

5. Simplify the expression \(1+ \frac{2}{x-3} \frac{x-3}{5+40 \left( \frac{x}{x^2-9} \right)}\).

(A) \(\frac{1}{5} \left( \frac{x+3}{2x+9} \right)\)     (B) \(\frac{x+3}{x-9}\)

(C) \(\frac{1}{5} \left( \frac{x+3}{x+9} \right)\)     (D) \(\frac{x+3}{2x-9}\)

(E) \(\frac{1}{5} \left( \frac{x-3}{x+9} \right)\)     (F) \(\frac{x-3}{x-9}\)

6. The shaded area in the figure is the complement of the sector of a circle of radius 6 inches lying inside the right triangle \(\Delta ABC\) with the angle \(\theta\) being expressed in radians. Express this shaded area as a function of \(S\), of \(\theta\).

\[
\text{Area of Sector} = \frac{1}{2} r^2 \theta
\]
\[
\text{Area of Right Triangle} = \frac{1}{2} bh = \frac{1}{2} \cdot 6 \cdot 6 \tan \theta
\]

\[
\text{So, Area of Shaded} = \frac{1}{2} (6)(\tan \theta) - \frac{1}{2} (6^2) \theta
\]
\[
= 18(\tan \theta - \theta)
\]

(A) \(S(\theta) = 36(\tan \theta - \theta)\)     (B) \(S(\theta) = 36(\sin \theta - \theta)\)

(C) \(S(\theta) = 18(\sin \theta - \theta)\)     (D) \(S(\theta) = 18(\cos \theta - \theta)\)

(E) \(S(\theta) = 18(\tan \theta - \theta)\)
7. Which of the following statements are true?
   I. The circle \((x-1)^2 + (y-2)^2 = 1\) has radius 1.
   II. The circle \((x-5)^2 + (y-6)^2 = 9\) has center \((6,5)\).
   III. The circle \((x-4)^2 + (y-4)^2 = 25\) has \(y\)-intercepts 1, 7.

   (A) I only     (B) II only     (C) I and III only     (D) III only     (E) II and III only

8. Find the area of the shaded region shown outside the square and inside the circle when the area of
the circle is \(25\pi\) sq. units.

   Area of Circle = \(\pi r^2\)
   \(\pi r^2 = 25\pi\)
   \(r = 5\)

   Area of Square = \((5\sqrt{2})^2\)
   = \(25 \times 2\)
   = 50

   Area of blue = \(25\pi - 50\)
   = \(29(\pi - 2)\)

   (A) \(5(4-\pi)\) sq. units  (B) \(5(\pi-1)\) sq. units  (C) \(25(\pi-2)\) sq. units
   (D) \(5(\pi-2)\) sq. units  (E) \(25(\pi-1)\) sq. units  (F) \(25(4-\pi)\) sq. units
9. Simplify the difference quotient \( \frac{f(x+h)-f(x)}{h} \), \( h \neq 0 \), when \( f(x) = 2x^2 - 4x - 4 \).
(A) \( 4x + 4 + 2h \)  (B) \( 4x - 4 + 2h \)  (C) \( 2x + 4 + 2h \)  (D) \( 2x - 4 + 2h \)  (E) \( 4x - 4 \)

\[
\frac{2(x+h)^2 - 4(x+h) - 4 - [2x^2 - 4x - 4]}{h} = \frac{2x^2 + 4xh + 4h^2 - 4x - 4h - 4 - 2x^2 + 4x}{h} = \frac{4h^2 + 4xh - 4h}{h} = 4h + 4x - 4
\]

10. Captain Calculus can leap over tall buildings. When he does so, his height \( s \) (in feet) off the ground after \( t \) seconds is given by \( s(t) = -t^2 + 7t + 34 \). For how many seconds is Captain Calculus more than 40 feet off the ground?
(A) 6 sec  (B) 9 sec  (C) 11 sec  (D) 5 sec  (E) 3 sec

\[
s(t) = -t^2 + 7t + 34 = 40
\]
\[
t^2 - 7t + 6 = 0
\]
\[
(t-1)(t-6) = 0
\]
\[
t = 1, t = 6
\]
So, he is above 40 ft for \( t = 6 \) = 5 seconds

11. If \( f(x) = 2x - 1 \) and \( g(x) = x + 3 \), which of the following gives \( f \circ g)(2) \)?
(A) 2  (B) 6  (C) 7  (D) 9  (E) 10

\[
f(g(2)) = f(5) = 9
\]
\[
f(2 + 3) = f(5) = 9
\]
\[
f(5) = 9
\]

12. Which of the following is a solution of the equation \( 2 - 3^x = -1 \)?
(A) \( x = -2 \)  (B) \( x = -1 \)  (C) \( x = 0 \)  (D) \( x = 1 \)  (E) No solution

\[
2 = 3^x - 1
\]
\[
3 = 3^x
\]
\[
\log_3 3 = \log_3 3^x
\]
\[
x = 1
\]
13. The length \( L \) of a rectangle is twice as long as its width \( W \). Which of the following gives the area \( A \) of the rectangle as a function of its width?

(A) \( A(W) = 3W \)  
(B) \( A(W) = \frac{1}{2} W^2 \)  
(C) \( A(W) = 2W^2 \)  
(D) \( A(W) = W^2 + 2W \)  
(E) \( A(W) = W^2 - 2W \)

\[
\begin{align*}
L &= 2W \\
\text{Area} &= 2W(W) \\
&= 2W^2
\end{align*}
\]

14. If \( p(x) = (x+2)(x+k) \) and if the remainder is 12 when \( p(x) \) is divided by \( x-1 \), then \( k = \)

(A) 2  
(B) 3  
(C) 6  
(D) 11  
(E) 13

So, by Remainder Theorem

\[
p(1) = 12 \\
(1+2)(1+k) = 12 \\
3(1+k) = 12 \\
1+k = 4 \\
k = 3
\]

or

\[
p(x) = x^2 + 2x + kx + 2k \\
= x^2 + (2+k)x + 2k \\
\text{by Synthetic Sub}
\]

\[
\begin{array}{c|c}
1 & 2+k & 2k \\
\hline
3+k & 3k & 9 \\
1 & 3+k & 3k+k
\end{array}
\]

So, \( 3+k = 12 \)  
\( 3k = 9 \)  
\( k = 3 \)

15. The set of all points \( (e^t, t) \), where \( t \) is a real number, is the graph of \( y = \)

(A) \( e^t \)  
(B) \( e^{1/t} \)  
(C) \( xe^{1/x} \)  
(D) \( \frac{1}{\ln x} \)  
(E) \( \ln x \)

So, inverse is \( (t, e^t) \)

\[
f^{-1} = y = e^x
\]

Find inverse of this inverse to get \( f(x) \)

\[
t = e^x \\
\ln t = \ln e^x \\
y = \ln t
\]

So, \( f(t) = \ln t \)

or \( \ln(t) \rightarrow x \)

16. If \( f(x) = \frac{4}{x-1} \) and \( g(x) = 2x \), then the solutions of \( f(g(x)) = g(f(x)) \) is

(A) \( \left\{ \frac{1}{3} \right\} \)  
(B) \( \{2\} \)  
(C) \( \{3\} \)  
(D) \( \{-1, 2\} \)  
(E) \( \left\{ \frac{1}{3}, 2 \right\} \)

\[
f(g(x)) = g(f(x)) \\
\frac{4}{2x-1} = 2\left(\frac{4}{x-1}\right) \\
\frac{1}{2x-1} = \frac{2}{x-1} \\
\frac{1}{2x-1} - \frac{2}{x-1} = 0 \\
(x-1) - 2(2x-1) = 0 \\
3x = 1 \\
x = \frac{1}{3}
\]
17. If the function $f$ is defined by $f(x) = x^5 - 1$, then $f^{-1}$, the inverse function of $f$, is defined by

$$f^{-1}(x) = \frac{1}{\sqrt[5]{x+1}}$$

(A) $\frac{1}{\sqrt[5]{x+1}}$  (B) $\frac{1}{\sqrt[3]{x+1}}$  (C) $\sqrt[5]{x-1}$  (D) $\sqrt[3]{x-1}$  (E) $\sqrt[5]{x+1}$

18. If $a$, $b$, $c$, $d$, and $e$ are real numbers and $a \neq 0$, then the polynomial equation $ax^5 + bx^3 + cx^3 + dx + e = 0$ has

(A) only one real root  (B) at least one real root  (C) an odd number of nonreal roots  (D) no real roots  (E) no positive real roots

Odd-degree functions have opposite end behaviors and are continuous. Hence, they must cross the $x$-axis at least once.

19. What are all values of $k$ for which the graph of $y = x^3 - 3x^2 + k$ will have three distinct $x$-intercepts?

(A) All $k > 0$  (B) All $k < 4$  (C) $k = 0, 4$  (D) $0 < k < 4$  (E) All $k$

If $k = 0$, then $x^3 - 3x^2 = 0$ and $x^2(x-3) = 0$, giving $x = 0, 3$. If $k > 0$, there will be three distinct $x$-intercepts. Based on the answer choices, $k > 0$.

20. If $f(g(x)) = x^3 + 3x^2 + 4x + 5$ and $g(x) = 5$, then $g(f(x)) = 5$

(A) $5x^2 + 15x + 25$  (B) $5x^3 + 15x^2 + 20x + 25$  (C) $1125$  (D) $225$  (E) $5$

If $g(x) = 5$, then $g(f(x)) = 5$.
21. If \( f(x) = 2x^3 + 4x^2 + Bx - 5 \) and if \( f(2) = 3 \) and \( f(-2) = -37 \), what is the value of \( A + B \)?

(A) -6  (B) -3  (C) -1  (D) 2  (E) It cannot be determined from the information given

\[
\begin{align*}
&\text{Plug in: } 16 + 4(3) + 2B - 5 = 3 \\
&\text{or } 16 - 12 - 5 + 2B = 3 \\
&\text{Solve: } A + B = -3 + 2 \\
&A = -3 \\
&B = 2
\end{align*}
\]

22. Suppose that \( f \) is a function that is defined for all real numbers. Which of the following conditions assures that \( f \) has an inverse function?

(A) The function \( f \) is periodic  
(B) The function \( f \) is symmetric with respect to the \( y \)-axis  
(C) The function \( f \) is concave up  
(D) The function \( f \) is a strictly increasing function  
(E) The function \( f \) is continuous  

passes horizontal & vertical line test.

23. If \( \log_a (2^a) = \frac{a}{4} \), then \( a = \)

(A) 2  (B) 4  (C) 8  (D) 16  (E) 32

\[
\begin{align*}
\log_a (2^a) &= \frac{a}{4} \\
a \cdot \log_a 2 &= \frac{a}{4} \\
\log_a 2 &= \frac{1}{4}, a \neq 0 \\
\log_a 2 &= \frac{1}{4} \\
a &= 2^{\frac{1}{4}} \\
&= \sqrt[4]{2}
\end{align*}
\]

24. If \( f(g(x)) = \ln(x^2 + 4) \), \( f(x) = \ln(x^2) \), and \( g(x) > 0 \) for all real \( x \), then \( g(x) = \)

(A) \( \frac{1}{\sqrt{x^2 + 4}} \)  
(B) \( \frac{1}{x^2 + 4} \)  
(C) \( \sqrt{x^2 + 4} \)  
(D) \( x^2 + 4 \)  
(E) \( x + 2 \)

\[
\begin{align*}
f(x) &= \ln(x^2) \\
f(g(x)) &= \ln(g(x)^2) \\
\therefore (g(x))^2 &= x^2 + 4 \\
g(x) &= \sqrt{x^2 + 4}
\end{align*}
\]
25. If \( \ln x - \ln \left( \frac{1}{x} \right) = 2 \), then \( x = \) \( x > 0 \)

(A) \( \frac{1}{e^2} \)  
(B) \( \frac{1}{e} \)  
(C) \( e \)  
(D) \( 2e \)  
(E) \( e^2 \)

\[ \ln \left( \frac{x}{\frac{1}{x}} \right) = z \]
\[ \ln x^2 = 2 \]
\[ 2 \ln x = z, \ x > 0 \]
\[ \ln x = 1 \]
\[ e^{\ln x} = e \]

26. If \( f(x) = \frac{x}{x+1} \), then the inverse function, \( f^{-1} \), is given by \( f^{-1}(x) = \)

(A) \( \frac{x-1}{x} \)  
(B) \( \frac{x+1}{x} \)  
(C) \( \frac{x}{1-x} \)  
(D) \( \frac{x}{x+1} \)  
(E) \( x \)

\[ y = \frac{x}{x+1} \]
\[ x = \frac{y}{y+1} \]
\[ (y+1)x = y \]
\[ xy + x = y \]
\[ xy - y = -x \]

27. If \( f(x) = e^x \sin x \), then the number of zeros of \( f \) on the closed interval \( [0, 2\pi] \) is

(A) 0  
(B) 1  
(C) 2  
(D) 3  
(E) 4

\[ e^x \sin x = 0 \]
\[ e^x = 0 \text{ or } \sin x = 0 \]
\[ \text{No Solution } \ x = 0, \pi, 2\pi \]

3 zeros

28. If \( h \) is the function given by \( h(x) = f(g(x)) \), where \( f(x) = 3x^2 - 1 \) and \( g(x) = |x| \), then \( h(x) = \)

(A) \( 3x^3 - |x| \)  
(B) \( |3x^2 - 1| \)  
(C) \( 3x^2|x| - 1 \)  
(D) \( 3|x| - 1 \)  
(E) \( 3x^2 - 1 \)

\[ h(x) = f(g(x)) \]
\[ = 3 \left( |x| \right)^2 - 1 \]
\[ = \begin{cases} 3(-x)^2 - 1 & \text{if } x < 0 \\ 3(x)^2 - 1 & \text{if } x \geq 0 \end{cases} \]

\[ = \begin{cases} 3x^2 - 1, & x < 0 \\ 3x^2 - 1, & x \geq 0 \end{cases} \]

\[ = 3x^2 - 1 \]
29. If \( e^{g(x)} = \frac{x^x}{x^2 - 1} \), then \( g(x) = \) 

(A) \( x \ln x - 2x \)  
(B) \( \frac{\ln x}{2} \)  
(C) \( (x - 2) \ln x \)  
(D) \( \frac{x \ln x}{\ln(x^2 - 1)} \)  
(E) \( x \ln x - \ln(x^2 - 1) \)

\[ e^{g(x)} = \frac{x^x}{x^2 - 1} \]
\[ \ln e^{g(x)} = \ln \left( \frac{x^x}{x^2 - 1} \right) \]
\[ g(x) = \ln x - \ln(x^2 - 1) \]

30. \( \frac{\ln(x^3e^x)}{x} = \)

(A) \( \frac{3(\ln x + e^x)}{x} \)  
(B) \( \ln(x^3 e^x - x) \)  
(C) \( \ln x^2 + 1 \)  
(D) \( \frac{3 \ln x + x}{x} \)  
(E) \( \frac{3 \ln x}{x} \)

31. If \( f(g(x)) = \sec(x^3 + 4) \), \( f(x) = \sec x^3 \), and \( g(x) \) is \textbf{not} an integer multiple of \( \frac{\pi}{2} \), then \( g(x) = \)

(A) \( \sqrt[3]{x+4} \)  
(B) \( \sqrt[3]{x-4} \)  
(C) \( \sqrt[3]{x^3+4} \)  
(D) \( \sqrt[3]{x-4} \)  
(E) \( \sqrt[3]{x+4} \)

\[ f(x) = \sec(x^3) \]
\[ f(g(x)) = \sec \left( (g(x))^3 \right) \]
\[ g(x) = -\sqrt[3]{x^3 + 4} \]

32. If \( f(x) = \log_b x \), then \( f(bx) = \)

(A) \( bf(x) \)  
(B) \( f(b)f(x) \)  
(C) \( 1 + f(x) \)  
(D) \( xf(b) \)  
(E) \( f(x) \)

\[ f(x) = \log_b x \]
\[ f(bx) = \log_b bx \]
\[ = \log_b b + \log_b x \]
\[ = 1 + \log_b x \]
\[ = 1 + f(x) \]
33. Which of the following statements is true?

(A) $\log_2 \frac{1}{2} > \log_2 \frac{1}{2}$

(B) $\log_3 (2 + 4) = \log_3 2 + \log_3 4$

(C) $\log_2 > \log 4$

(D) $\log \left( \sqrt{3} \right) = \frac{2}{3}$

(E) $\log \left( \frac{1}{4} \right) = \frac{2}{2}$

None of these are true.