Worksheet 2.1—Tangent Line Problem

Show all work. No calculator permitted, except when stated.

Short Answer

1. Find the derivative function, \( f'(x) \), for each of the following using the limit definition.

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

(a) \( f(x) = 2x^2 + 3x - 4 \)

\[
f'(x) = \lim_{h \to 0} \frac{(2(x+h)^2 + 3(x+h) - 4) - (2x^2 + 3x - 4)}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 3x + 3h - 2x^2 - 3x - 4}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{4x + 2h + 3}{h}
\]

\[f'(x) = 4x + 3\]

(b) \( f(x) = \frac{3}{x-1} \)

\[
f'(x) = \lim_{h \to 0} \frac{\frac{3}{x+h-1} - \frac{3}{x-1}}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{3(x-h) - 3(x-h)(x-1)}{h(x+h-1)(x-1)}
\]

\[
f'(x) = \lim_{h \to 0} \frac{3x - 3x - 3h + 3x - 3h + 3}{h(x+h-1)(x-1)}
\]

\[
f'(x) = \lim_{h \to 0} \frac{3x-3-3x+3}{h(x+h-1)(x-1)}
\]

\[f'(x) = \frac{3}{(x-1)^2}\]

(c) \( f(x) = \sqrt{x-2} \)

\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h-2} - \sqrt{x-2}}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{\frac{(x+h-2)-(x-2)}{h(x+h-2+x-2)}}{h}
\]

\[
f'(x) = \lim_{h \to 0} \frac{x+h-2-x+2}{h(x+h-2+x-2)}
\]

\[
f'(x) = \lim_{h \to 0} \frac{h}{h(x+h-2+x-2)}
\]

\[f'(x) = \frac{1}{\sqrt{x-2}} + \frac{1}{\sqrt{x-2}}\]

\[f'(x) = \frac{1}{\sqrt{x-2}}\]

2. Find the slope of the tangent lines to the graphs of the following functions at the indicated points. Use the alternate form.

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

(a) \( f(x) = 3 - 2x \) at \((-1,5)\)

\[
f'(-1) = \lim_{x \to -1} \frac{(3-2x) - (5)}{x - (-1)}
\]

\[
= \lim_{x \to -1} \frac{-2-2x}{x + 1}
\]

\[f'(-1) = -2\]

(b) \( g(x) = 5 - x^2 \) at \( x = 2 \)

\[
g'(2) = \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2}
\]

\[
= \lim_{x \to 2} \frac{(5-x^2) - (5-2^2)}{x - 2}
\]

\[
= \lim_{x \to 2} \frac{-x^2 + 9}{x - 2}
\]

\[
= \lim_{x \to 2} \frac{(x-3)(x+3)}{(x-2)}
\]

\[
= \lim_{x \to 2} \frac{(x-3)}{x-2}
\]

\[= -2\]

\[g'(2) = -4\]
3. Find the equation of the tangent line, in Taylor Form: \( y = y_1 + m(x - x_1) \), for \( g(x) = x^2 + 1 \) at \( (2, 5) \).

Use the modified form to find \( g'(2) \).

\[
g(x) = x^2 + 1 \quad \text{at} \quad x = 2
\]

\[
g'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h} = \frac{4 + 4h + h^2 + 1 - 4}{h} = \frac{h(4 + h)}{h} = 4
\]

\[
f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}
\]

\[
g'(2) = 4
\]

4. Find the equation of the tangent line, in Taylor Form: \( y = y_1 + m(x - x_1) \), for \( y = \sqrt{x} - 1 \) at \( x = 9 \). Use the alternate form to find \( y'(9) \).

\[
y'(9) = \lim_{h \to 0} \frac{y(9 + h) - y(9)}{h}
\]

\[
y'(9) = \frac{\sqrt{9 + h} - 1}{h}
\]

\[
y'(9) = 2 \text{ avg. value at } x = 9
\]

\[
y'(9) = \frac{\sqrt{9 + h} - 1}{h} = \frac{3}{2}(x^2 + 3)
\]

\[
y'(9) = \frac{3}{2} \text{ slope at } x = 9
\]

\[
y'(9) = \frac{3}{2} (x - 9)
\]
5. Find an equation of the line that is tangent to \( f(x) = x^3 \) and parallel to the line \( 3x - y + 1 = 0 \). Remember, parallel lines have the same slope, but different base camps.

\[
\begin{align*}
3y - x + 1 &= 0 \\
y &= 3x + 1 \\
\text{slopes} &= 3
\end{align*}
\]

\[
\begin{align*}
f(x) &= x^3 \\
f'(x) &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{(x+h)^3 - x^3}{h} \\
&= \frac{3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= 3x^2 + 3x + 1
\end{align*}
\]

So, \( f'(x) = 3x^2 + 3x + 1 \) parallel slope.

\[
\begin{align*}
x &= 1 \\
m &= 3 \\
(1, f(1)) &\in (1, 1) \\
(1, 1) &\in (-1, -1)
\end{align*}
\]

Find an equation of the line that is tangent to \( f(x) = x^3 \) and parallel to the line \( 3x - y + 1 = 0 \). Remember, parallel lines have the same slope, but different base camps.

6. Find the equations of the two lines, \( l_1 \) and \( l_2 \), that are tangent to the graph of \( f(x) = x^2 \) if each pass through the point \( (1, -3) \), as shown at right. Hint: equate two different expressions for finding the slope of a line. Solve the resulting equation.

\[
\begin{align*}
\text{ALGEBRA SLOPE} \\
p &= (x_1, x_1^2), (1, -3) \\
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{x_2^2 - (-3)}{x_2 - 1} \\
&= \frac{x_2 + 3}{x_2 - 1}
\end{align*}
\]

\[
\begin{align*}
\text{CALCULUS SLOPE} \\
f(x) &= x^2 \\
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= 2x
\end{align*}
\]

So, \( m = f'(x) \)

\[
\begin{align*}
x^2 + 3 &= 2x \quad \text{(1)} \\
x^2 + 3 &= 2x^2 - 2x \quad \text{(2)} \\
x = x^2 - 2x - 3 \quad \text{(3)} \\
(x-3)(x+1) &= 0 \\
x = 3 \quad \text{or} \quad x = -1
\end{align*}
\]

So, the 2 points of tangency are \((3, 9)\) and \((-1, -1)\)

\[
\begin{align*}
\text{Equations of lines} \\
e_{(3, 9)}: m = f'(3) = 2(3) = 6 \\
l_1: y = g + 6(x - 3) \quad \text{(4)} \\
e_{(-1, -1)}: m = f'(-1) = 2(-1) = -2 \\
l_2: y = 1 - 2(x + 1) \quad \text{(5)}
\end{align*}
\]
7. The graph of a function \( f(x) \) is show above. For which value(s) of \( x \) is the graph of \( f(x) \) not differentiable. In each case, explain why not.

\[
\text{\( f(x) \) is not differentiable at:}
\]
- \( x = -2 \), because \( f(x) \) is not continuous at \( x = -2 \) (\( f_{\text{left}}(\text{at}) = -\infty \))
- \( x = -1 \), because the graph of \( f(x) \) has a cusp at \( x = -1 \) (slopes are different on either side of \( x = -1 \)).
- \( x = 0 \), because \( f(x) \) has a vertical tangent at \( x = 0 \) (slope is infinite at \( x = 0 \))
- \( x = 2 \), because \( f(x) \) is not continuous at \( x = 2 \) (\( f(2) \) is undefined)

8. For each of the following, the limit represents \( f'(c) \) for a function \( f(x) \) and a number \( x = c \). Find both \( f \) and \( c \).

(a) \[ \lim_{h \to 0} \frac{5-3(1+h)}{h} - 2 = f'(1) \]

\[
\begin{align*}
\lim_{h \to 0} \frac{5-3(1+h)}{h} &= \frac{f(c+h) - f(c)}{h} \\
&= f'(c) \\
&= 5-3(1+1) \\
&= 5 - 3c \\
&= 5 - 3(1) \\
&= 5 - 3 \\
&= 2
\end{align*}
\]

so, \( f(x) = 5 - 3x \) and \( c = 1 \)

(b) \[ \lim_{h \to 0} \frac{(2+h)^2 + 8}{h} = f'(-2) \]

\[
\begin{align*}
\lim_{h \to 0} \frac{(2+h)^2 + 8}{h} &= \frac{f(c+h) - f(c)}{h} \\
&= f'(c) \\
&= (2+h)^2 + 8 \\
&= 2^2 + h^2 + 8 \\
&= 4 + h^2 + 8 \\
&= h^2 + 12 \\
&= 0
\end{align*}
\]

so, \( f(x) = x^2 \) and \( c = -2 \)

(c) \[ \lim_{x \to 6} \frac{-x^2 + 36}{x - 6} = f'(6) \]

\[
\begin{align*}
\lim_{x \to 6} \frac{-x^2 + 36}{x - 6} &= \frac{f(x+h) - f(x)}{h} \\
&= f'(x) \\
&= -x^2 + 36 \\
&= -6^2 + 36 \\
&= 0
\end{align*}
\]

so, \( f(x) = -x^2 \) and \( c = 6 \)

(d) \[ \lim_{x \to 9} \frac{2\sqrt{x} - 6}{x - 9} = f'(9) \]

\[
\begin{align*}
\lim_{x \to 9} \frac{2\sqrt{x} - 6}{x - 9} &= \frac{f(x+h) - f(x)}{h} \\
&= f'(x) \\
&= 2\sqrt{x} - 6 \\
&= 2\sqrt{9} - 6 \\
&= 2(3) - 6 \\
&= 6 - 6 \\
&= 0
\end{align*}
\]

so, \( f(x) = 2\sqrt{x} \) and \( c = 9 \)
9. Using the alternate form, determine whether each of the following function is differentiable at the indicated point. Show the work that leads to your answer.

(a) \( f(x) = \begin{cases} 
5 - 4x, & x \leq 0 \\
-2x^2, & x > 0 
\end{cases} \) at \( x = 0 \)

\[
\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{(5 - 4x) - 5}{x - 0} = \lim_{x \to 0^-} \frac{-4x}{x} = \lim_{x \to 0^-} -4 = -4
\]

\[
\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{-2x^2 - 5}{x} = \lim_{x \to 0^+} \frac{-2x^2}{x} - \lim_{x \to 0^+} \frac{5}{x} = \text{DNE or } -\infty
\]

Since the slopes from the right of \( x = 0 \) are approaching \( -\infty \), \( f(x) \) is not differentiable at \( x = 0 \).

(b) \( f(x) = \begin{cases} 
(x-1)^3, & x \leq 1 \\
(x-1)^2, & x > 1 
\end{cases} \) at \( x = 1 \)

\[
\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{(x-1)^3 - 0}{x - 1} = \lim_{x \to 1^-} \frac{(x-1)^2}{1} = \lim_{x \to 1^-} ((x-1)(x-1)) = \lim_{x \to 1^-} (x-1) = 0
\]

\[
\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(x-1)^2 - 0}{x - 1} = \lim_{x \to 1^+} \frac{(x-1)(x-1)}{x - 1} = \lim_{x \to 1^+} (x-1) = 0
\]

Since 0 = 0, \( f(x) \) is differentiable at \( x = 1 \).

10. True or False. If false, explain why or give a counterexample.

(a) The slope of the tangent line to the differentiable function \( f \) at the point \( (2, f(2)) \) is

\[
\frac{f(2+h) - f(2)}{h}
\]

False, it is \( \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \) the given expression is missing the limit.

(b) If a function is continuous at a point, then that function is differentiable at that point.

False, it might be differentiable, but no guarantees. \( \text{Ex: } f(x) = x^{1/3} \text{ at } x = 0 \)

True: if a function is differentiable, then it is continuous.

(c) If a function’s slopes from both the right and the left at a point are the same, then that function is differentiable at that point.

False, could be, but not always. It must also be continuous.

\( \text{Ex: } f(x) = \begin{cases} 2x+1, & x \leq 0 \\
2x + 2, & x > 0 
\end{cases} \) at \( x = 0 \)

(d) If a function is differentiable at a point, then that function is continuous at that point.

True!
11. Using your calculator to zoom in, determine if \( h(x) = \sqrt{x^2 + 0.0001} + 0.99 \) is locally linear at \( x = 0 \). Give a reason for your answer.

\[
yes, \text{ it is, and so it is also differentiable at } x = 0. \text{ After zooming in around } x = 0 \text{ many, many times, the graph becomes linear (at first, though, it looks like } y = |x| + 1. \]

**Multiple Choice**

12. A function will fail to be differentiable at all of the following except

(A) A vertical asymptote  \quad (B) A removable discontinuity \quad (C) A cusp

(D) A vertical tangent line  \quad (E) A horizontal tangent line

\[
f'(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2}, & x \neq 2 \\
1, & x = 2 
\end{cases}
\]

13. Let \( f \) be the function defined above. Which of the following statements about \( f \) are true?

I. \( \lim_{x \to 2} f(x) \) exists

II. \( f \) is continuous at \( x = 2 \)

III. \( f \) is differentiable at \( x = 2 \)

(A) I only  \quad (B) II only  \quad (C) III only  \quad (D) I and II only  \quad (E) I, II, and III

14. Let \( f \) be a differentiable function such that \( f(2) = 1 \) and \( f'(2) = 4 \). Let \( T(x) \) be the equation of the tangent line to \( f(x) \) at \( x = 2 \). What is the value of \( T(1.9) \)?

\[
\begin{align*}
T(x) & = y_1 + m(x - x_1) \\
T(x) & = f(2) + f'(2)(x - 2) \\
T(x) & = 1 + 4(x - 2) \\
T(1.9) & = 1 + 4(1.9 - 2) \\
& = 1 - 0.4 \\
T(1.9) & = 0.6
\end{align*}
\]
15. Let \( f \) be a function such that \( \lim_{{h \to 0}} \frac{f(7+h) - f(7)}{h} = 5 \). Which of the following must be true?

I. \( f \) is continuous at \( x = 7 \)
II. \( f \) is differentiable at \( x = 7 \)
III. The derivative of \( f \) is differentiable at \( x = 7 \)

(A) I only          (B) II only           (C) I and II only          (D) I and III only          (E) II and III only

16. At \( x = 4 \), the function given by 
\[
 h(x) = \begin{cases} 
  x^2, & x \leq 4 \\
  4x, & x > 4 
\end{cases}
\]

is

(A) Undefined
(B) Continuous but not differentiable
(C) Differentiable but not continuous
(D) Neither continuous nor differentiable
(E) Both continuous and differentiable