Worksheet 2.5—Rates of Change and Particle Motion I

Show all work. No calculator unless otherwise stated.

Short Answer

1. Let $E(x)$ be the elevation, in feet, of the Mississippi River $x$ miles from its headwaters at Lake Itasca in Park Rapids, Minnesota, for $0 \leq x \leq 2320$.

   (a) What are the units of $E'(x)$? f/ mil

   (b) What is the sign (positive or negative) of $E'(x)$? Why?

   (c) (Calculator Permitted) If $E(0) = 1475$, $E(2320) = 0$, to 3-decimals, find the average change in elevation, in feet per mile, of the Mississippi River for $0 \leq x \leq 2320$. Show the work that leads to your answer. Using correct units, explain the meaning of your answer.

   $\frac{0 - 1475}{2320 - 0} \text{ f/mile} = -0.635 \text{ f/mile}$

2. An economist is interested in how the price of a graphing calculator affects its sales. Suppose that at a price of $p$ dollars, a quantity, $q$, calculators are sold, then the quantity of calculators sold is a function of the calculator’s price, that is, if $q = f(p)$.

   (a) In a complete sentence using correct units, explain the meaning of $f(150) = 20,000$.

      At a price of $150, 20,000$ calculators are sold.

   (b) In a complete sentence using correct units, explain the meaning of $f'(150) = -50$

      At a price of $150, the number of calculators sold is DECREASING by 50 calculators per dollar.

   (c) Assuming the rate from part (b) holds for $150 \leq p \leq 170$, how many calculators are predicted to sell when the price of a calculator is $160$? Show the work that leads to your answer.

      $\text{Calculators sold at } \$160 = 20,000 + \text{ change in calculators sold due to } \$10/\text{calc increase}$

      $= 20,000 + (-50 \text{ calc/}\$)(\$10)$

      $= 20,000 - 500$

      $= 19,500$ calculators sold.
3. Suppose $M(t)$ is the amount of time, in minutes, it takes to boil a cup of water if the water has an initial temperature of $t$, degrees Fahrenheit. In a complete sentence, with units, explain the meaning of each of the following:

(a) $M(50) = 19$
   
   When the water is $50\, ^\circ F$, it takes 19 minutes for that water to boil.

(b) $M^{-1}(19) = 50$
   
   When it takes 19 minutes to boil water, the water’s initial temperature was $50\, ^\circ F$.

(c) $M'(50) = 3$
   
   When the water is $50\, ^\circ F$, the time to boil the water is increasing at 3 minutes per degree Fahrenheit.

(d) $\left(M^{-1}\right)(18) = 0.2$
   
   At the 18th minute when the water comes to a boil, the temperature of the water is increasing by 0.2 degrees Fahrenheit per minute.

4. If $g(v)$ is the fuel efficiency, in miles per gallon, of a car going at $v$ miles per hour, $g(v) = \text{mpg}/\text{mph}$.

(a) What are the units of $g'(v)$? miles per gallon per miles per hour or $\text{mpg}/\text{mph}$.

(b) In a complete sentence with units, what is the practical meaning of the statement $g'(55) = -0.54$?
   
   At 55 miles per hour, fuel efficiency is decreasing by 0.54 miles per gallon for each mph increase.

5. Suppose that $C(T)$ is the cost of heating your house, in dollars per day, when the temperature outside is $T$ degrees Fahrenheit.

(a) Explain in practical terms the meaning of $C'(23) = -0.21$.
   
   When the outdoor temperature is 23\, ^\circ F, the cost to heat your house is decreasing by 21 cents per day.

(b) If $C(23) = 2.94$ and $C'(23) = -0.21$, approximate the cost to heat your house when the temperature is 20 degrees, $C(20)$. Show the work that leads to your answer.

   Let $-0.21\, ^\circ F$ be the rate to heat your house for $20\, ^\circ F \leq T \leq 23\, ^\circ F$.

   $\frac{C(23) - C(20)}{23 - 20} \approx -0.21$ dollars/day per $^\circ F$

   $\frac{2.94 - C(20)}{3} \approx -0.63$ dollars/day per $^\circ F$

   $\frac{2.94 - C(20)}{3} \approx -0.63$ dollars/day per $^\circ F$

   $\frac{C(20) - C(22)}{3} \approx 3.57$ dollars/day per $^\circ F$

   $\approx \# \text{3.57 per day}$
6. If \( x(t) \) represents the position of a particle along the \( x \)-axis at any time, \( t \), fill in the blanks in the statements below with the best answer so that they become true facts (not opinions).

(a) “Initially” means when \( \frac{\text{time}}{x} = 0 \).

(b) “At the origin” means \( \frac{x}{v} = 0 \).

(c) “At rest” means \( \frac{\text{velocity}}{x} = 0 \).

(d) If the velocity of the particle is positive, then the particle is moving to the \( \text{right} \).

(e) If the velocity of the particle is \( \text{negative} \), then the particle is moving to the left.

(f) To find average velocity over a time interval, divide the change in \( \text{position} \) by the change in time.

(g) \( \text{instantaneous velocity} \) is the velocity at a single moment (instant) in time.

(h) If the acceleration of the particle is positive, then the \( \text{velocity} \) is increasing.

(i) If the acceleration of the particle is \( \text{negative} \), then the velocity is decreasing.

(j) In order for a particle to change directions, the \( \text{velocity} \) must change signs.

(k) One way to determine \( \text{total distance travelled} \) over a time interval, when given the position function or graph, is to find the sum of the absolute values of the differences in position between all resting points.

7. If the position of a particle along a horizontal line is given by \( x(t) = x^2 + x - 6 \) for \( 0 \leq t \leq 3 \)

(a) Sketch the graph of the particle’s position on the given interval.

\[ x(t) = x^2 + x - 6 = 0 \]
\[ (x+3)(x-2) = 0 \]
\[ x = -3, x = 2 \]

(b) What is the particle’s displacement on the given interval? Show the work that leads to your answer.

\[ \text{Displacement} = x(3) - x(0) = 6 - (-6) = 12 \]

(c) Find the total distance traveled by the particle on the given interval. Show the work that leads to your answer.

\[ \text{Distance} = |x(3) - x(0)| = |6 - (-6)| = |12| = 12 \]

\( \text{Displacement and Distance Traveled are only the same because } x(t) \text{ is monotonic increasing } (x' = 0) \text{ on the entire interval } t \in [0,3]. \)
8. The data in the table below gives selected values for the velocity, in meters/minute, of a particle moving along the x-axis. The velocity \( v \) is a differentiable function of time \( t \).

<table>
<thead>
<tr>
<th>Time ( t ) (min)</th>
<th>0</th>
<th>2</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity ( v(t) ) (meters/min)</td>
<td>-3</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>5</td>
</tr>
</tbody>
</table>

(a) At \( t = 0 \), is the particle moving to the right or to the left? Justify.
\[
(0) = -3 < 0, \text{ so at } t = 0, \text{ the particle is moving to the } \text{LEFT} \text{ at a rate of 3 meters per minute.}
\]

(b) Is there a time during the time interval \( 0 \leq t \leq 12 \) minutes when the particle is at rest? Explain your answer.

Since \( v(t) \) is differentiable, it is also continuous. 
\[
&\text{Since } v(0) = -3 < 0 < 2 = v(2), \text{ by the } \text{IVT, there is a time } t \in (0, 2) \text{ where } v(t) = 0 \text{ and the particle is at rest.}
\]

(c) Use data from the table to find an approximation for \( v'(10) \) and explain the meaning of \( v'(10) \) in terms of the motion of the particle. Show the computations that lead to your answer, and indicate units of measure.
\[
8 < 10 < 12, \text{ so } v'(t) \approx \frac{v(12) - v(8)}{12 - 8} = \frac{5 - 7}{12 - 8} = -\frac{2}{4} = -\frac{1}{2} \text{ meters/min}.
\]
At \( t = 10 \text{ min} \), the particle’s velocity is \text{DECREASING by approximately} \( \frac{1}{2} \text{ meter/min} \text{ per minute.}

(d) Find the average acceleration of the particle for \( 8 \leq t \leq 12 \) min. Explain what this number means in terms of the particle’s velocity on that interval.
\[
\text{Avg Accel} = \frac{v(12) - v(8)}{12 - 8} = \frac{5 - 7}{12 - 8} = -\frac{1}{2} \text{ meters/min}^2.
\]
On the interval from \( t = 8 \text{ min} \) to \( t = 12 \text{ min} \), the particle’s acceleration, on average, was \( -\frac{1}{2} \text{ meters/min}^2 \) (velocity was decreasing, on average, at a rate of \( \frac{1}{2} \text{ meter/min} \text{ per minute.}

(e) Let \( a(t) \) denote the acceleration of the particle at time \( t \), such that \( v'(t) = a(t) \). Is there guaranteed to be a time \( t = c \) in the interval \( 0 \leq t \leq 12 \) such that \( a(c) = 0 \)? Justify your answer.

Since \( v(t) \) is differentiable, \( v'(t) = a(t) \) is continuous.
\[
&\text{Since } \frac{v(12) - v(8)}{12 - 8} = \frac{5 - 7}{12 - 8} = -\frac{1}{2} > 0
\]
\& \[
\frac{v(8) - v(6)}{8 - 6} = \frac{2 - 5}{2} = -\frac{3}{2} < 0,
\]
there must be a time \( t \in (6, 12) \text{ min} \) when \( a(t) = 0 \). (IVT for slopes.)
9. The graph above represents the velocity \( v \), in feet per second, of a particle moving along the \( x \)-axis over the time interval for \( 0 \leq t \leq 9 \) seconds.

(a) At \( t = 4 \) seconds, is the particle moving to the right or left? Justify.
   \( \text{To the right because } v(4) > 0. \)

(b) At what time(s) is the particle at rest? Justify.
   \( \text{At } t = 5 \text{ sec and } t = 7 \text{ sec because } v(t) = 0 \text{ and } v(9) = 0. \)

(c) At what time(s) does the particle change direction? Justify.
   \( \text{At } t = 7 \text{ sec, since } v(t) \text{ changes from positive to negative at } t = 7 \text{ sec.} \)

(d) On what open intervals \( 0 < t < 9 \) is the particle moving left? Justify.
   \( \text{from } 7 < t < 9, \text{ because on this interval } v(t) < 0. \)

(e) What is the acceleration of the particle at \( t = 4 \) seconds? Show the work that leads to your answer.
   \[ a(t) = \frac{dv}{dt} = -12 \text{ ft/sec} \text{ or } -6 \text{ ft/sec}^2. \]

(f) On what open intervals \( 0 < t < 9 \) is the acceleration of the particle positive? Justify.
   \( \text{This happens for } 0 < t < 3 \text{ seconds and } t \in (5, 6) \text{ seconds.} \)

(g) What is the average acceleration of the particle over the interval \( t \in [3, 6] \) seconds? Show the computations that lead to your answer, and indicate units of measure.
   \[ \text{Average acceleration } = \frac{v(6) - v(3)}{6 - 3} = \frac{-12 - (-10)}{3} = -\frac{2}{3} \text{ ft/sec}^2 \]

(h) On what open intervals \( 0 < t < 9 \) is the speed of the particle decreasing? Justify.
   \( \text{Speed is decreasing when } v(t) \text{ and } v'(t) = a(t) \text{ are opposite signs (i.e., when graph of } v(t) \text{ is heading towards the } t \text{-axis).} \)

(i) Without knowing the initial position of the particle, is it still possible to determine the time at which the particle is farthest right for \( 0 \leq t \leq 9 \)? If not, explain. If so, find this value of \( t \), and explain.
   \( \text{Yes, the particle is farthest right at } t = 7 \text{ seconds, since } v(t) > 0 \text{ for all } t \in [0, 7] \text{ seconds, and } v(t) < 0 \text{ for } t \in (7, 9] \text{ see.} \)
10. A particle moves along the x-axis so that at time \( t \geq 0 \), its position is given by \( x(t) = t^3 - 3t^2 - 9t + 2 \).

(a) At \( t = 0 \), is the particle moving to the right or to the left? Justify.

\[
x(t) = t^3 - 3t^2 - 9t + 2
\]
\[
x'(t) = 3t^2 - 6t - 9
\]
\[
v(0) = -9 < 0,
\]
\[
\text{so, at } t = 0, \text{ the particle is moving to the left.}
\]

(b) At what time(s) does the particle change directions. Justify.

\[
v(t) = 3t^2 - 6t - 9 = 0
\]
\[
3(t-3)(t+1) = 0
\]
\[
\text{so, } t = 3 \text{ or } t = -1
\]
\[
\text{since } v(t) \text{ changes from negative to positive at } t = 3, \text{ the particle changes direction (left to right)}
\]
\[
\text{at } t = 3.
\]

(c) At \( t = 1/2 \), is the velocity of the particle increasing or decreasing? Explain your answer.

\[
v(t) = 3t^2 - 6t - 9
\]
\[
v'(t) = a(t) = 6t - 6
\]
\[
a(\frac{1}{2}) = 6(\frac{1}{2}) - 6 = 3 - 6 = -3 < 0
\]
\[
\text{velocity is DECREASING.}
\]

(d) At \( t = 1/2 \), is the speed of the particle increasing or decreasing? Explain your answer.

\[
v'(t) = a(t) = 6t - 6
\]
\[
\text{from (c), } a(\frac{1}{2}) = -3 < 0
\]
\[
\text{so, since } v(t) < 0 \text{ & } a(t) < 0 \text{ (same sign),}
\]
\[
\text{at } t = \frac{1}{2}, \text{ the particle's speed is INCREASING!}
\]

(e) Find all values of \( t \) for which the particle is moving to the left.

\[
v(t) = 3t^2 - 6t - 9 < 0
\]
\[
\text{for all } t \in [0, 3)
\]

(f) What is the particle’s acceleration at \( t = \frac{1}{3} \)? Explain, with units, the meaning of your answer in terms of the particle’s velocity.

\[
a(t) = 6t - 6
\]
\[
a(\frac{1}{3}) = 6(\frac{1}{3}) - 6 = 2 - 6 = -4
\]
\[
\text{since } a(\frac{1}{3}) = -4 < 0,
\]
\[
\text{at } t = \frac{1}{3} \text{ the particle’s velocity is DECREASING by } 4 \text{ distance/time units per unit of time.}
\]

11. Fill in the blanks so that each statement below is true.

(a) If velocity is negative and acceleration is positive, then speed is \underline{decreasing}.

(b) If velocity is positive and speed is decreasing, then acceleration is \underline{negative}.

(c) If velocity is positive and decreasing, then speed is \underline{decreasing}.

(d) If speed is increasing and acceleration is negative, then velocity is \underline{negative}.

(e) If velocity is negative and increasing, then speed is \underline{decreasing}.

(f) If the particle is moving to the left and speed is decreasing, then acceleration is \underline{positive}. 
12. The graph above shows the velocity, \( v(t) \), in miles per hour of a particle moving along the x-axis for \( 0 \leq t \leq 11 \) hours. It consists of a semi circle and two line segments. Use the graph and your knowledge of motion to answer the following questions.

(a) At what time, \( 0 \leq t \leq 11 \) hours, is the speed of the particle the greatest?

**Justify.** Speed is greatest when graph of \( v(t) \) is furthest away from the t-axis in either direction. This occurs at \( t = 8 \) hrs.

(b) At which of the times, \( t = 2 \), \( t = 6 \), or \( t = 9 \) hours, is the acceleration of the particle greatest?

**Justify.** Acceleration is greatest when the slopes of the graph of \( v(t) \) are greatest. Of the listed times, the slope is greatest at \( t = 9 \).

(c) Over what open time interval(s) \( 0 < t < 11 \) hours is the particle moving to the left? Justify.

**Justify.** Particle moves left when the graph of \( v(t) < 0 \). This occurs for \( t \in (4, 10) \) hours.

(d) Over what open time interval(s) \( 0 < t < 11 \) hours is the velocity of the particle increasing? Justify.

**Justify.** Velocity is increasing when the slopes of the graph of \( v(t) \) are positive. This occurs for \( t \in (0, 2) \cup (8, 10) \) hours.

(e) Over what open time interval(s) \( 0 < t < 11 \) hours is the speed of the particle increasing? Justify.

**Justify.** Speed is increasing when the graph of \( v(t) \) is moving AWAY from the t-axis in either direction. This occurs for \( t \in (0, 2) \cup (4, 8) \cup (10, 11) \) hrs.

(f) At what times on \( 0 < t < 11 \) is the acceleration of the particle undefined?

**Justify.** Acceleration undefined when slopes of graph of \( v(t) \) are undefined. This occurs at \( t = 4 \) and \( t = 8 \) hrs.

(g) Find the area of the semicircle on the interval \( 0 \leq t \leq 4 \) bounded by the curve and the x-axis, then find the area of the triangle on the interval \( 4 \leq t \leq 10 \) bounded by the curve and the x-axis, and finally, find the area of the triangle on the interval \( 10 \leq t \leq 11 \) bounded by the curve and the x-axis. If all of these areas were positive and added together, propose what quantity this might be in terms of the particle’s movement on \( 0 \leq t \leq 11 \) hours.

\[
\left( \frac{\pi}{2} + \frac{\sqrt{15}}{2} \right) \text{ miles}
\]

This gives the total distance travelled, in miles, over the 11-hour period from \( t = 0 \) to \( t = 11 \) hrs.
Multiple Choice

13. A violent storm system is in the Central Texas area is moving in a straight line through New Braunfels. The Weather Service Office forecasts that \( t \) hours after mid-day, the storm will be a distance of \( s(t) = 12 + t - t^2 \) miles from New Braunfels. How fast will the storm be moving at 2:00 pm, and in what direction, in relation to NB, will it be moving, respectively?

(A) 10 mph, towards NB
(B) 5 mph, towards NB
(C) 10 mph, away from NB
(D) 3 mph, towards NB
(E) 3 mph, away from NB

14. (Calculator Permitted) It took Mr. Wenzel from noon until 7 pm to drive from NB to his in-laws’ house North of Dallas 385 miles away. After \( t \) hours of driving, his distance from NB was given in miles by

\[
s(t) = \frac{165}{7} t^2 - \frac{110}{49} t^3.
\]

(i) What was his average speed for the trip to his in-laws’?

(A) 53 mph (B) 54 mph (C) 55 mph (D) 56 mph (E) 57 mph (F) 58 mph

(ii) (Calculator Permitted) Mr. Wenzel’s instantaneous speed twice coincided with his average speed. At what time did it first happen?

(A) 1:33 pm (B) 1:28 pm (C) 5:41 pm (D) 5:31 pm (E) 1:38 pm

15. (Calculator Permitted) If a rock is thrown vertically upwards from the surface of the planet Newton with an initial velocity of 11 ft/sec, its height after \( t \) seconds is given by \( h = 11t - \frac{1}{2}t^2 \). Find the velocity of the rock after it has risen 56 feet.

(A) -1 ft/sec (B) 2 ft/sec (C) 0 ft/sec (D) 1 ft/sec (E) 3 ft/sec

16. (Calculator Permitted) A particle moves along a straight line with velocity given by

\[ v(t) = 4 - (0.98)^{-t^2} \] at time \( t \geq 0 \). What is the acceleration of the particle at time \( t = 4 \)?

(A) -0.223 (B) 2.618 (C) 8.284 (D) 0.010 (E) -0.092

\[ v'(4) = -0.223 \]