① $f(x) = x\sqrt{6-x}$, $D_f: \{x \mid x \le 6\}$

$f'(x) = (6-x)^{1/2} + \frac{1}{2}x(6-x)^{-1/2}(-1)$

$= (6-x)^{-1/2}\left(6 - x - \frac{1}{2}x\right)$

$f'(x) = \dfrac{6 - \frac{3}{2}x}{\sqrt{6-x}}$, $D_{f'}: \{x \mid x < 6\}$

So f is continuous on [0,6] and diff'able on (0,6) and f(0) = 0 = f(6), so Rolle's Thm Applies.

$f'(x) = \dfrac{6 - \frac{3}{2}x}{\sqrt{6-x}} = 0$

when $\boxed{x = 4}$ $\boxed{D}$

③ $f(x) = x^3 - x - 1$ is a polynomial and is continuous & diff'able for all $x \in \mathbb{R}$, so MVT applies on [-1, 2].

$f'(x) = \dfrac{f(2) - f(-1)}{2 - (-1)}$

$3x^2 - 1 = \dfrac{5 - (-1)}{3}$

$\boxed{3x^2 - 1 = 2}$

$3x^2 = 3$

$x = -1$ or 1

Must be on the OPEN interval

but $x = -1 \notin (-1, 2)$

so $\boxed{x = 1}$ $\boxed{E}$

⑤ MVT apply?

I. $f(x) = \frac{1}{x+1}$ on [0,2], YES!
f(x) is not continuous or diff'able at $x = -1 \notin [0,2]$ or $(0,2)$

II. $f(x) = x^{1/3}$ on [0,1], YES!
f(x) is continuous $\forall x \in \mathbb{R}$ and not diff'able at $x = 0 \notin (0,1)$.

III. $f(x) = |x|$ on [-1,1], No!
f(x) is not diff'able at $x = 0 \in (-1,1)$

so only $\boxed{\text{I, II apply}}$ $\boxed{B}$

② for $f'(c) = 0$ on (-1,1), Rolle's Thm must Apply.

So I must be true of all f(x). ✓

II. $f(x) = (\cos x)^3$, $f'(x) = 3\cos^2 x(-\sin x)$
is continuous and diff'able $\forall x \in \mathbb{R}$ ✓
$f(-1) = f(1)$ since $\cos x$ is an even function ✓

III. $f(x) = |\sin \pi x| \to$ continuous $\forall x \in \mathbb{R}$

$f(x) = \sqrt{(\sin \pi x)^2}$

$f'(x) = \frac{1}{2}\left((\sin \pi x)^2\right)^{-1/2} \cdot 2\sin \pi x \cdot \cos \pi x \cdot \pi$

$= \dfrac{\pi \sin \pi x \cdot \cos \pi x}{|\sin \pi x|}$

f is not diff'able when $\pi x = 0 + \pi n, n \in \mathbb{Z}$
$x = 0 + n, n \in \mathbb{Z}$

but it IS diff'able between integers
& $|\sin(-\pi x)| = |\sin \pi x|$ since $\sin x$ is odd ✓

So $\boxed{G}$ all of I, II, III are true.

④ $f(x) = x + x^{2/3}(1-x)^{1/3} \to$ continuous $\forall x \in \mathbb{R}$

$f'(x) = 1 + \frac{2}{3}x^{-1/3}(1-x)^{1/3} + x^{2/3}\left(\frac{1}{3}(1-x)^{-2/3}(-1)\right)$

$= 1 + \frac{1}{3}x^{-1/3}(1-x)^{-2/3}\left[2(1-x) - x\right]$

$f'(x) = 1 + \dfrac{2 - 3x}{3\sqrt[3]{x} \cdot \sqrt[3]{(1-x)^2}}$, $x \ne 0, 1$

So f is not diff'able at the endpoints, but IS diff'able on (0,1), so MVT applies!! so,

$f'(x) = \dfrac{f(1) - f(0)}{1 - 0}$

$\cancel{1} + \dfrac{2 - 3x}{3\sqrt[3]{x} \cdot \sqrt[3]{(1-x)^2}} = \dfrac{1 - 0}{1} = \cancel{1}$

$\dfrac{2 - 3x}{3\sqrt[3]{x} \cdot \sqrt[3]{(1-x)^2}} = 0$

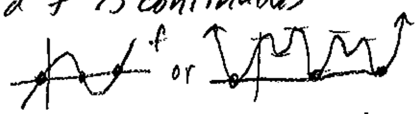
when $2 - 3x = 0$

$3x = 2$

$\boxed{x = \frac{2}{3}} \in (0,1)$

$\boxed{E}$

(6) f is twice diffable so
f is continuous and f' is continuous
f has 3 roots



* So f(x) can have more than 3 horizontal tangents, so I is not necessarily true.
* The curvature will have to change at least once, so $f''(x) = 0$ at least once.
II is true. **B**

(7) Jenna's avg velocity is $\frac{53 \text{ miles}}{48/60 \text{ hrs}}$ = **66.25 mph** her instantaneous velocity, by MVT, must be 66.25 mph, so set the speed limit to **C**, **65 mph** the highest speed limit given that is less than 66.25 mph.

(8) (a) $f(x) = \cos 2x$, $[-\frac{\pi}{12}, \frac{\pi}{6}]$

Rolle's does <u>NOT</u> apply. f(x) is continuous on $[-\frac{\pi}{12}, \frac{\pi}{6}]$ and diff'able on $(-\frac{\pi}{12}, \frac{\pi}{6})$, but

$f(\frac{\pi}{6}) = \frac{1}{2} \neq \frac{\sqrt{3}}{2} = f(-\frac{\pi}{12})$

(b) $g(x) = \frac{x^2 - 2x - 3}{x + 2}$ on $[-1, 3]$

$g(3) = 0$, $g(-1) = 0$

and g(x) is continuous and diff'able $\forall x \neq -2 \in [-1, 3]$

so, <u>YES</u> Rolle's Applies

(9) MVT applies??

(a) $f(x) = \ln(x-1)$ on $[2, 4]$

$D_f: x > 1$ and is continuous & diffable $\forall x > 1$, so <u>YES</u>!

MVT applies: $\frac{1}{x-1} = \frac{\ln 3}{2}$

$x - 1 = \frac{2}{\ln 3}$, $\boxed{x = 1 + \frac{2}{\ln 3}}$

(b) $f(x) = \begin{cases} \arcsin x, & -1 \le x < 1 \\ \frac{1}{2}x, & 1 \le x \le 3 \end{cases}$ on $[-1, 3]$

continuous @ x = 1?

$\lim_{x \to 1^-} f(x) = \frac{\pi}{2} \neq \frac{1}{2} = \lim_{x \to 1^+} f(x)$, so f(x) is <u>NOT</u> continuous at $x = 1 \in [-1, 3]$, so <u>NO</u>, MVT does not apply.

(c) $g(x) = \frac{x+1}{x}$ on $[\frac{1}{2}, 2]$

g(x) is continuous and diff'able $\forall x \neq 0 \notin [\frac{1}{2}, 2]$, so <u>YES</u>, MVT applies.

$g(x) = 1 + \frac{1}{x}$, $\quad \frac{\frac{3}{2} - 3}{2 - 1/2}$

$g'(x) = -\frac{1}{x^2} = \frac{\frac{3}{2} - 3}{2 - 1/2}$

$-x^2 = \frac{3/2}{-3/2}$

$x^2 = 1$

~~$x = -1$~~ or $\boxed{x = 1}$

(-1) not in $(\frac{1}{2}, 2)$

(d) $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$

Yes, f(x) is continuous & diff'able $\forall x \in [0, \pi]$

$f'(x) = \frac{f(\pi) - f(0)}{\pi - 0}$

$2\cos x + 2\cos 2x = 0$

$\cos x + \cos 2x = 0$

$\cos x + \cos^2 x - (\sin^2 x) = 0$ *double angle ID

$\cos x + \cos^2 x - (1 - \cos^2 x) = 0$ *Pythagorean ID

$2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = \frac{1}{2}$ or $\cos x = -1$

$\boxed{x = \frac{\pi}{3}}$ or ~~$x = \frac{5\pi}{3}$~~ or ~~$x = \pi$~~

$\frac{5\pi}{3}, \pi \notin (0, \pi)$

(10) $f(x) = -x^4 + 4x^3 + 8x^2 + 5$, on [0,5]

MVT applies since f(x) is continuous & diff'able $\forall x \in \mathbb{R}$.

$$f'(x) = \frac{f(5) - f(0)}{5 - 0}$$ [slope of secant line (and tangent line(s))]

$$-4x^3 + 12x^2 + 16x = 15$$

$$-4x^3 + 12x^2 + 16x - 15 = 0$$

$x = 0.673 = A$ (store as A)

$x = 3.793 = B$ (store as B)

$f(A) = 9.646 = C$ (store as C)

$f(B) = 131.402 = D$ (store as D)

tangent line eqs: $y = C + 15(x - A)$, $y = D + 15(x - B)$

Notes (for calculator)

* put f(x) into Y1 on calculator, turn graph off for now.

* From Homescreen, find Avg rate of change by: (Y1(5) - Y1(0))/(5 - 0)

* solve eq: put $Y2 = -4x^3 + 12x^2 + 16x - 15$, $Y3 = 0$

* Find point of intersections: [2nd] [trace] [#5] on window X[0,5], Y[-5,5]

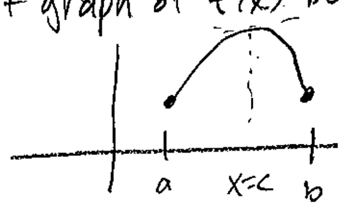
* find function values from homescreen Y1(A), Y1(B), Y1(5), Y1(0)

$f(0) = 5$

secant line eq

$y = 5 + 15(x - 0)$ or $y = 5 + 15x$

(11) Let graph of f(x) be



Rolle's applies on
* [a,b]
* $f'(c) = 0$

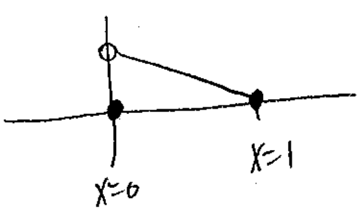
(a) $g(x) = f(x) + k$
Vertical shift k units. This doesn't change x-locations, so Rolle's Applies on
* [a,b]
* $f'(c) = 0$

(b) $g(x) = f(x - k)$
Horizontal shift k units.
Rolle's Applies on
* [a+k, b+k]
* $f'(c+k) = 0$

(c) $g(x) = kf(x)$
Vertical dilation by factor of k, same as (a)

(d) $g(x) = f(kx)$
Horz dilation
$k > 1 \to$ compress
$0 < k < 1 \to$ stretch
Rolle's Applies on
* $[\frac{a}{k}, \frac{b}{k}]$
* $f'(\frac{c}{k}) = 0$

(12) $f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \le 1 \end{cases}$



* Rolle's Does NOT apply since f(x) is not continuous at $x = 0 \in [0,1]$.

(13) $f(x) = \begin{cases} 1, & x = 0 \\ ax + b, & 0 < x \le 1 \\ x^2 + 4x + c, & 1 < x \le 3 \end{cases}$

For MVT to apply, f needs to be continuous on [0,3] and diffable on (0,3)

* continuity at x=0:
$\lim_{x \to 0^+} f(x) = f(0)$
$a(0) + b = 1$
$b = 1$

continuity at x=1:
$\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$
$a(1) + b = 1^2 + 4(1) + c$
$a + 1 = 5 + c$
$a = 4 + c$

* diff'able at x=1:
$f'(x) = \begin{cases} a, & x < 1 \\ 2x + 4, & x > 1 \end{cases}$
$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x)$
$a = 2 + 4$
$a = 6$

so $c = a - 4$
$c = 6 - 4$
$c = 2$

so $a = 6, b = 1, c = 2$

(14) f(x) is continuous & diffable on [6,15]
$f(6) = -2$, $f'(x) \le 10 \ \forall x \in [6,15]$
What is Max value of $f(15)$?

By MVT, $f'(x) = \frac{f(15)-f(6)}{15-6}$ for some x

So $\frac{f(15)-f(6)}{15-6} \le 10$

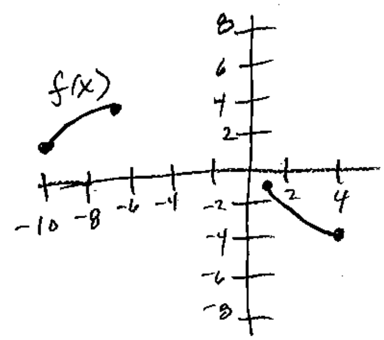
$f(15) - (-2) \le 10(9)$

$f(15) \le 90 - 2$, So $f(15) \le 88$

So f(15) is A MAX of 88

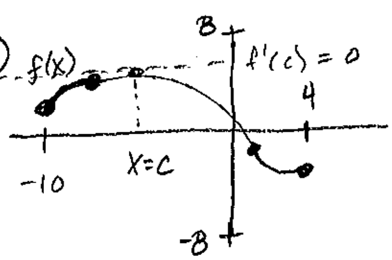
(15) f is ~~diff'able~~ on [-10,4], f' is also continuous (So f is twice-differentiable).
and f is continuous

(a) since $f(-10) > 0$ and $f(4) < 0$,
since $f(4) < 0 < f(-10)$,
by IVT, f(x) must have a zero
on [-10,4]

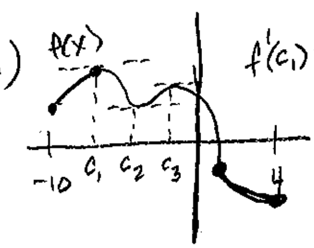


(b) Since f' is continuous, and since
$f'(-8) > 0$ and $f'(3) < 0$, by IVT,
Since $f'(-8) < 0 < f'(3)$, f' must equal zero
on [-8,3] which is a subset of [-10,4].

(c)



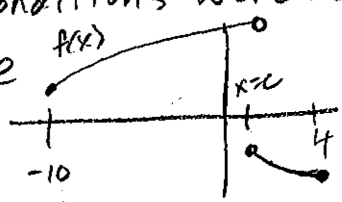
(d)



$f'(c_1) = f'(c_2) = f'(c_3) = 0$

GRAPHS WILL VARY

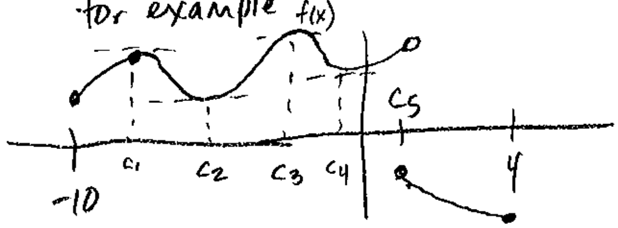
(e) The conditions were necessary for parts (a) & (b) lest the graph look like



which is not continuous nor diff'able at $x = c \in [-10,4]$.
* Notice the graph of f(x) has neither an x-intercept nor a hori~~zontal~~ ~~tangent~~.

The conditions were NOT necessary to meet the requirements of (c) & (d).

for example



* Notice the graph of f(x) is not continuous nor diff'able at $x = c_5 \in [-10,4]$, still no zeros of f(x), but we still have at least 2 horizontal asymptotes.