**Worksheet 6.6—Improper Integrals**
Show all work. No calculator unless explicitly stated.

**Short Answer**
1. Classify each of the integrals as proper or improper integrals. Give a clear reason for each.

   - (a) \[ \int_{5}^{\infty} \frac{dx}{(x-2)^2} \]
     - Improper, \[ \infty \] as upper limit of integration

   - (b) \[ \int_{1}^{5} \frac{dx}{(x-2)^2} \]
     - Improper, \( x=2 \in [1, 5] \), the interval of integration

   - (c) \[ \int_{2}^{5} \frac{dx}{(x-2)^2} \]
     - Improper, \( \forall x \neq 2 \notin [3, 5] \), outside the interval of integration

   - (d) \[ \int_{3}^{5} \frac{dx}{(x-2)^2} \]
     - Proper.

2. Answer the following.

   - (a) If \( \int_{a}^{\infty} f(x)dx = K \) and \( 0 < g(x) \leq f(x) \), what can we say about \( \int_{a}^{\infty} g(x)dx \)?
     - Since integral of \( f(x) \) converges, by comparison, anything smaller than it, over the same interval, will converge too. Since \( g(x) \leq f(x) \), \( \int_{a}^{\infty} g(x)dx \) converges (but not necessarily to \( K \)).

   - (b) If \( \int_{a}^{\infty} f(x)dx = K \) and \( 0 < f(x) < g(x) \), what can we say about \( \int_{a}^{\infty} g(x)dx \)?
     - If the integral of \( f(x) \) converges to \( K \), we cannot say anything for sure about an integral over the same interval of a larger function. Since \( g(x) > f(x) \), \( \int_{a}^{\infty} g(x)dx \) may either converge or diverge.

   - (c) If \( \int_{a}^{\infty} f(x)dx \) diverges and \( 0 < f(x) \leq g(x) \), what can we say about \( \int_{a}^{\infty} g(x)dx \)?
     - If the integral of \( f(x) \) diverges, then any integral over the same interval of a larger function will diverge as well.

   - (d) If \( \int_{a}^{\infty} f(x)dx \) diverges and \( 0 < g(x) < f(x) \), what can we say about \( \int_{a}^{\infty} g(x)dx \)?
     - If the integral of the larger function diverges, then the integral of the smaller function on the same interval may diverge or converge, so we cannot say anything conclusively.
3. If \( \int_{1}^{\infty} \frac{1}{x^p} \, dx \) converges for \( p > 1 \), what can be said in general about improper integrals of the form \( \int_{a}^{\infty} \frac{1}{x^p} \, dx \)? For what values of \( a \) does the function diverge? Converge? To what?

For \( p > 1 \), \( a > 0 \), but not necessarily to \( \frac{1}{p-1} \). We still call these integrals \( \int_{a}^{\infty} \frac{1}{x^p} \, dx \), where \( a > 0 \), \( p \)-series integrals.

If \( p \leq 1 \), the integral diverges, as the graph of \( y = \frac{1}{x^p} \) does not go toward the \( x \)-axis (horizontal asymptote) fast enough (steep enough) to cause the integral to converge to a finite value.

4. Determine if the improper integral converges or diverges by finding a function to compare it to. Justify by showing the inequality and discussing the convergence/divergence of the function to which you compare.

(a) \( \int_{2}^{\infty} \frac{x^5}{x^6 - 1} \, dx \)

(b) \( \int_{1}^{\infty} \frac{x^3 + 1}{(x^4 + x^3 + 1)^2} \, dx \)

(c) \( \int_{1}^{\infty} \frac{dx}{(x + 5)^3} \)

(d) \( \int_{4}^{\infty} \frac{3 + \sin x}{x} \, dx \)

Multiple Choice

5. \( \int_{0}^{\infty} x^2 e^{-x^3} \, dx = \)

(A) \( -\frac{1}{3} \)  (B) 0  (C) \( \frac{1}{3} \)  (D) 1  (E) Diverges
6. Which of the following gives the value of the integral \( \int_1^{1.01} x^{-1.01} \, dx \)?

(A) 1 \hspace{1cm} (B) 10 \hspace{1cm} (C) 100 \hspace{1cm} (D) 1000 \hspace{1cm} (E) Diverges

\[
\int_1^{1.01} x^{-1.01} \, dx = \frac{1}{x^{1.01}-1} \bigg|_1^{1.01} = \frac{1}{1.01-1} = 100
\]

7. Which of the following gives the value of the integral \( \int_0^{1/2} x^{-0.5} \, dx \)?

(A) 1 \hspace{1cm} (B) 2 \hspace{1cm} (C) 3 \hspace{1cm} (D) 4 \hspace{1cm} (E) Diverges

\[
\int_0^{1/2} \frac{1}{\sqrt{x}} \, dx = 2 \left[ \sqrt{b} - \sqrt{a} \right]_{a=0}^{b=1/2} = 2 \left( \frac{1}{2} - 0 \right) = 1
\]

8. Which of the following gives the value of the integral \( \int_0^1 \frac{1}{x-1} \, dx \)?

(A) \(-1\) \hspace{1cm} (B) \(-1/2\) \hspace{1cm} (C) 0 \hspace{1cm} (D) 1 \hspace{1cm} (E) Diverges

**Method 1:**
\[
\int_0^1 \frac{1}{x-1} \, dx = \text{Intuition} \times \frac{1}{x-1} \sim \frac{1}{x}
\]
which diverges as we approach BOTH the Horz Asympt. & the Vert Asympt., so DIVERGES

\[
\int_0^1 \frac{1}{x-1} \, dx = \ln |x-1| \bigg|_0^1 = -\infty
\]

**Method 2:**
\[
\int_0^1 \frac{1}{x-1} \, dx = \text{Intuition} \times \frac{1}{x-1} \sim \frac{1}{x}
\]
which diverges as we approach BOTH the Horz Asympt. & the Vert Asympt., so DIVERGES

\[
\int_0^1 \frac{1}{x-1} \, dx = \ln |x-1| \bigg|_0^1 = -\infty
\]
9. Which of the following gives the value of the area under the curve $y = \frac{1}{x^2 + 1}$ in the first quadrant?

(A) $\frac{\pi}{4}$  
(B) 1  
(C) $\frac{\pi}{2}$  
(D) $\pi$  
(E) Diverges

10. Determine if $\int_{0}^{2} f(x) \, dx$ is convergent or divergent when $f(x) = \begin{cases} x^{-1/2}, & x \leq 1 \\ x, & 1 < x \leq 2 \end{cases}$, and if it is convergent, find its value.

(A) $\frac{1}{2}$  
(B) $\frac{5}{2}$  
(C) $\frac{7}{2}$  
(D) 4  
(E) Diverges

11. $\int_{0}^{\infty} \frac{x}{2} \sqrt{\frac{3}{x^2 - 2}} \, dx =$

(A) $\frac{3 \cdot 2^{2/3}}{4}$  
(B) $2^{2/3}$  
(C) $-\frac{3 \cdot 2^{2/3}}{4}$  
(D) $-\frac{3 \cdot 2^{2/3}}{2}$  
(E) Diverges
Free Response

12. (AP 1996-1) Consider the graph of the function \( h(x) = e^{-x^2} \) for \( 0 \leq x < \infty \).

(a) Let \( R \) be the unbounded region in the first quadrant below the graph of \( h \). Find the volume of the solid generated when \( R \) is revolved about the \( y \)-axis.

\[
\begin{align*}
V &= 2\pi \int_0^\infty (x \times e^{-x^2}) \, dx \\
&= 2\pi \int_0^\infty e^{-x^2} \, dx \\
&= \frac{\sqrt{\pi}}{2} \\
&= \frac{\pi}{2} \\
&= \frac{\pi}{2} - \pi \left[ e^x - 1 \right] \\
&= \frac{\pi}{2}
\end{align*}
\]

(b) Let \( A(w) \) be the area of the shaded rectangle shown in the figure. Show that \( A(w) \) has its maximum value when \( w \) is the \( x \)-coordinate of the point of inflection of the graph of \( h \).

Area of rectangle = \( A(w) = w \cdot e^{-w^2} \)

\[
\begin{align*}
A'(w) &= 1 \cdot e^{-w^2} + w(-2w \cdot e^{-w^2}) = 0 \\
A(w) &= e^{-w^2} \left( 1 - 2w^2 \right) = 0 \\
e^{-w^2} = 0 \quad \text{or} \quad 1 - 2w^2 = 0 \\
\text{No Solution} \quad \text{or} \quad w^2 = \frac{1}{2}
\end{align*}
\]

\( \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \)

Note: \( \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} \)

Since \( A'(w) > 0 \) \( \forall w \in \left[ 0, \sqrt{\frac{1}{2}} \right) \)

\( A'(w) < 0 \) \( \forall w > \sqrt{\frac{1}{2}} \),

\( w = \sqrt{\frac{1}{2}} \) maximizes \( A(w) \)

Absolutely on \( w \in [0, \infty) \)

\[
\begin{align*}
h(x) &= e^{-x^2} \\
h'(x) &= -2xe^{-x^2} \\
h''(x) &= -2e^{-x^2} + 4x^2e^{-x^2} = 0 \\
&= -2e^{-x^2}(1 - x^2) = 0
\end{align*}
\]

No solution \( x = \sqrt{\frac{1}{2}} \)

So \( b \) has an inflection pt.

at \( x = \sqrt{\frac{1}{2}} \) which is the value that maximizes \( A(w) \).
13. (AP 2001-5) Let \( f \) be the function satisfying \( f'(x) = -3xf(x) \), for all real numbers \( x \), with \( f(1) = 4 \) and \( \lim_{x \to \infty} f(x) = 0 \).

(a) Evaluate \( \int_{-\infty}^{\infty} -3xf(x) \, dx \). Show the work that leads to your answer.

(b) Use Euler’s method, starting at \( x_0 = 1 \) with a step size of 0.5, to approximate \( f(2) \).

(c) Write an expression for \( y = f(x) \) by solving the differential equation \( \frac{dy}{dx} = -3xy \) with the initial condition \( f(1) = 4 \).
14. (AP 2010B-5) Let \( f \) and \( g \) be the functions defined by \( f(x) = \frac{1}{x} \) and \( g(x) = \frac{4x}{1 + 4x^2} \), for all \( x > 0 \).

(a) Find the absolute maximum value of \( g \) on the open interval \((0, \infty)\) if the maximum exists. Find the absolute minimum value of \( g \) on the open interval \((0, \infty)\) if the minimum exists. Justify your answers.

\[
\begin{align*}
g(x) &= \frac{4x}{1 + 4x^2} \\
g'(x) &= \frac{(1 + 4x^2)(4) - (4x)(8x)}{(1 + 4x^2)^2} \\
g'(x) &= \frac{4(1 + 4x^2) - 8x^2}{(1 + 4x^2)^2} \\
g'(x) &= \frac{4(1 - 4x^2)}{(1 + 4x^2)^2} \\
g''(x) &= \text{DNE when } 4(1 - 4x^2) = 0 \\
&= \frac{1}{2} \\
x &= \pm \frac{\sqrt{2}}{2} \\
x &= 0
\end{align*}
\]

Since \( g' > 0 \) for \( x \in (0, \frac{\sqrt{2}}{2}) \) and \( g' < 0 \) for \( x > \frac{\sqrt{2}}{2} \), \( g \) has an absolute maximum at \( x = \frac{\sqrt{2}}{2} \). Thus, the maximum value is \( g\left(\frac{\sqrt{2}}{2}\right) = \frac{4\left(\frac{\sqrt{2}}{2}\right)}{1 + 4\left(\frac{\sqrt{2}}{2}\right)^2} = \frac{2}{1 + 1} = 1 \).

\( g \) has no minimum values on \( x \in (0, \infty) \).

(b) Find the area of the unbounded region in the first quadrant to the right of the vertical line \( x = 1 \), below the graph of \( f \), and above the graph of \( g \).

\[
\text{Area} = \int_1^\infty \left( f(x) - g(x) \right) \, dx
\]

\[
= \int_b^\infty \left( f(x) - g(x) \right) \, dx
\]

\[
= \left[ \ln \left( \frac{b}{\sqrt{1 + 4b^2}} \right) + \frac{1}{2} \ln 5 \right]_b^\infty
\]

\[
= \ln \left( \frac{\sqrt{b^2 + 1}}{b} \right) + \frac{1}{2} \ln 5
\]

\[
= \ln \left( \frac{b}{\sqrt{1 + 4b^2}} \right) + \frac{1}{2} \ln 5
\]

\[
= \frac{1}{2} \left[ \ln \left( \frac{b}{\sqrt{b^2 + 1}} \right) + \ln 5 \right]
\]

\[
= \frac{1}{2} \left[ \ln \left( \frac{b}{\sqrt{b^2}} \right) + \ln 5 \right]
\]