Worksheet 7.2 II—Parametric & Vector Review

Show all work on a separate sheet of paper. A calculator IS permitted, except on problems 1 & 2.

1. (No Calculator) The position of a particle at any time \( t \geq 0 \) is given by \( x(t) = t^2 - 2, \ y(t) = \frac{2}{3} t^3 \).

(a) Find the magnitude of the velocity vector at \( t = 2 \).

\[
\begin{align*}
\vec{x}(t) &= t^2 - 2 \\
\vec{y}(t) &= \frac{2}{3} t^3 \\
\vec{x}'(t) &= 2t \\
\vec{y}'(t) &= 2t^2 \quad \text{(b)} \\
\vec{v}(t) &= \vec{v}(2) = 4 \hat{i} + 8 \hat{j} \\
||\vec{v}(t)|| &= \sqrt{6^2 + 8^2} \\
&= 10 \\
&= \sqrt{100} \\
&= \sqrt{10} \\
&= 4\sqrt{5}.
\end{align*}
\]

(b) Set up an integral expression to find the total distance traveled by the particle from \( t = 0 \) to \( t = 4 \).

\[
\text{Distance} = s = \int_0^4 \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} \, dt
\]

(c) Find \( \frac{dy}{dx} \) as a function of \( x \).

\[
\begin{align*}
\frac{dy}{dx} &= \frac{dx}{dy} \\
&= \frac{\frac{dy}{dt}}{\frac{dx}{dt} + 2t} \\
&= \frac{2t^2}{x} \\
&= \frac{2t^2}{t^2 - 2} \\
&= \frac{2t^2}{t^2 - 2} \quad \text{(given)} \\
&= \frac{2(2)}{2} \\
&= \frac{4}{2} \\
&= 2
\end{align*}
\]

(d) At what time \( t \) is the particle on the \( y \)-axis? Find the acceleration vector at this time.

\[
\begin{align*}
\text{On the \( y \)-axis means} \ y = 0 \\
\text{So,} \ (x = \frac{2}{3} t^3 = 0) \quad \text{or} \quad t = 0 \\
\text{So, particle is on the \( y \)-axis at} \ t = 0.
\end{align*}
\]

So, particle is on the \( y \)-axis at \( t = \sqrt{2} \).

\[
\begin{align*}
x'(t) &= 2t \\
y'(t) &= \frac{2}{3} t^3 \\
x''(t) &= 2 \\
y''(t) &= 4t
\end{align*}
\]

So, \( \vec{a}(t) = \langle x'(t), y'(t) \rangle \)

\[
\begin{align*}
\vec{a}(\sqrt{2}) &= \langle 2, 4\sqrt{2} \rangle \quad \text{(c)}
\end{align*}
\]
2. (No Calculator) An object moving along a curve in the $xy$-plane has position $\langle x(t), y(t) \rangle$ at time $t$ with the velocity vector $\vec{v}(t) = \left( \frac{1}{t+1}, \frac{2t}{x} \right)$. At time $t = 1$, the object is at $(\ln 2, 4)$.

(a) Find the position vector.

$$\hat{s}(t) = \langle x(t), y(t) \rangle + \int_1^t \left( \frac{1}{x+1}, \frac{2t}{x} \right) \, dx$$

$$= \langle \ln 2 + \ln |x(t)|^2, 4 + x^2 \rangle$$

$$= \langle \ln |x(t)|^2 + \ln 2 - \ln 2, 4 + 1 \rangle = \langle x(t) |t+1, 3 \rangle$$

(b) Write an equation for the line tangent to the curve when $t = 1$.

$$p_t = (\ln 2, 4)$$

$$\frac{dx}{dt} \bigg|_{t=1} = \frac{y'(t)}{x'(t)} = \frac{\frac{2t}{1+1}}{\frac{1}{t+1}} = \frac{2}{1} = 4$$

$$y = y + 4(x - \ln 2)$$

(c) Find the magnitude of the velocity vector when $t = 1$.

$$||\vec{v}(t)|| = \sqrt{\left( \frac{x'(t)}{x'(t)} \right)^2 + \left( \frac{y'(t)}{y'(t)} \right)^2}$$

$$= \sqrt{\left( \frac{2}{1} \right)^2 + (2)^2}$$

$$= \sqrt{16 + 4} = \sqrt{20}$$

(d) At what time $t > 0$ does the line tangent to the particle at $\langle x(t), y(t) \rangle$ have a slope of 12?

$$\frac{dx}{dt} = \frac{\frac{2t}{1+1}}{\frac{1}{t+1}} = 12$$

$$\frac{2}{1} = 12$$

$$= \frac{2(t+1)}{t+1} = 12$$

$$t = 2$$

So, tangent line has slope of 12 at $t = 2$.

3. A particle moving along a curve in the $xy$-plane has position $\langle x(t), y(t) \rangle$, with $x(t) = 2t + 3\sin t$ and $y(t) = t^2 + 2\cos t$, where $0 \leq t \leq 10$. Find the velocity vector at the time when the particle’s vertical position is $y = 7$.

$\begin{align*}
\text{Vert} & \quad \text{position} = 7 \\
\text{y}(8) & = 7 \\
t^2 + 2\cos t & = 7 \\
t^2 + 2\cos 8 & = 7 \\
t^2 + 2(0.14) & = 7 \\
t^2 & = 6.72 \\
t & = \sqrt{6.72} \\
& \approx 2.59
\end{align*}$
4. A particle moving along a curve in the $xy$-plane has position $\langle x(t), y(t) \rangle$ at time $t$ with
$$ y_1 = \frac{dx}{dt} = 1 + \sin(t^3). $$ The derivative $\frac{dy}{dt}$ is not explicitly given. For any $t \geq 0$, the line tangent to the curve at $\langle x(t), y(t) \rangle$ has a slope of $t + 3$. Find the acceleration vector of the object at time $t = 2$.

$$ a_x = \frac{dy}{dt} \left( \frac{d^2y}{dt^2} \right) = t + 3 $$

$$ a_y = \frac{d^2y}{dt^2} = y_2 $$

5. An object moving along a curve in the $xy$-plane has position $\langle x(t), y(t) \rangle$ at time $t$ with $\frac{dx}{dt} = \cos(e^t) = y$ and $\frac{dy}{dt} = \sin(e^t)$ for $0 \leq t \leq 2$. At time $t = 1$, the object is at the point $(3, 2)$.

(a) Find the equation of the tangent line to the curve at the point where $t = 1$.

$$ \frac{dy}{dx} \bigg|_{t=1} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin(e^t)}{\cos(e^t)} $$

At $t = 1$, the point is $(3, 2)$ (given).

$$ \frac{dy}{dx} = -0.450 = \Delta y \quad \text{(given)} $$

So, the equation is $y = 2 - 0.450(x - 3)$.

(b) Find the speed of the object at $t = 1$.

$$ \left\| \frac{dy}{dt} \right\| = \sqrt{\left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right)^2 + \left( \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \right)^2} $$

So, the speed is $\frac{dy}{dt} = \sqrt{\frac{dy}{dx}^2 + \frac{dy}{dx}^2}$

(c) Find the total distance traveled by the object over the time interval $0 \leq t \leq 2$.

$$ \text{Distance} = L = \int_0^2 \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt $$

So, the distance is $L = 2$.

(d) Find the position of the object at time $t = 2$.

$$ \vec{r}(2) = \langle x(1) + \int_1^2 x'(t) \, dt, y(1) + \int_1^2 y'(t) \, dt \rangle $$

$$ = \langle 3 + \int_1^2 \cos(e^t) \, dt, 2 + \int_1^2 \sin(e^t) \, dt \rangle $$

So, the position is $\vec{r}(2) = \langle 2.695, 1.675 \rangle$. 
6. A particle moving along a curve in the $xy$-plane has position $(x(t), y(t))$ at time $t$ with

$$\frac{dx}{dt} = \sin(t^3 - t)$$
and
$$\frac{dy}{dt} = \cos(t^3 - t).$$

At time $t = 3$, the particle is at the point $(1, 4)$.

(a) Find the acceleration vector for the particle at $t = 3$.

$$\mathbf{a}(3) = \left( x'(3), y'(3) \right) = \left( 1.02, 23.542 \right).$$

(b) Find the equation of the tangent line to the curve at the point where $t = 3$.

$$\frac{dx}{dt} = \frac{y'(3)}{x'(3)} = -0.466 \Rightarrow y = 4 - 0.466(x - 1).$$

(c) Find the magnitude of the velocity vector at $t = 3$.

$$||v(3)|| = \sqrt{(x'(3))^2 + (y'(3))^2} = 1$$

(Pythagorean identity again)

(d) Find the position of the particle at time $t = 2$.

$$s(t) = \left( x(3) + \int_2^3 x'(t) \, dt, \ y(3) + \int_2^3 y'(t) \, dt \right)$$

$$= \left( 1 + \int_2^3 1 \, dt, \ 4 + \int_2^3 1 \, dt \right)$$

$$= \left( 4.002, 4.002 \right).$$

7. An object moving along a curve in the $xy$-plane has position $(x(t), y(t))$ at time $t$ with

$$\frac{dy}{dt} = 2 + \sin(e^t).$$

The derivative of $\frac{dx}{dt}$ is not explicitly given. At $t = 3$, the object is at the point $(4, 5) = (x(e^3), y(e^3))$.

(a) Find the $y$-coordinate of the position at time $t = 1$.

$$y(1) = y(e^3) + \int_1^3 y'(t) \, dt$$

$$= 5 + \int_1^3 1 \, dt$$

$$= 1.26 \Rightarrow A \text{ (store)}$$

(b) At time $t = 3$, the value of $\frac{dy}{dt}$ is $-1.8$. Find the value of $\frac{dx}{dt}$ when $t = 3$.

$$\frac{dx}{dt} \bigg|_{t=3} = \frac{y'(3)}{x'(3)} = -1.8$$

$$x'(3) = \frac{y'(3)}{-1.8}$$

$$\frac{dx}{dt} \bigg|_{t=3} = x'(3) = -1.8 \Rightarrow B \text{ (store)}$$

(c) Find the speed of the object at time $t = 3$.

$$\text{Speed at } t = 3 \text{ is } ||v(3)|| = \sqrt{(x'(3))^2 + (y'(3))^2}$$

$$= \sqrt{3^2 + 1.8^2} = 3.36 \Rightarrow B \text{ (store)}$$