Worksheet 8.2—Polar Area
Show all work. **Calculator permitted** except unless specifically stated.

**Short Answer:** Sketch a graph, shade the region, and find the area.

1. one petal of \( r = 2 \cos(3\theta) \)

2. one petal of \( r = 4 \sin(2\theta) \)

3. interior of \( r = 2 + 2 \cos \theta \)

   (no calculator)

4. interior of \( r = 2 - \sin \theta \)

   (no calculator)
5. interior of $r^2 = 4\sin(2\theta)$

\[
\text{Area} = \frac{1}{2} \int_0^{\pi/2} r^2 \, d\theta
\]

\[
= \frac{1}{2} \int_0^{\pi/2} 4\sin 2\theta \, d\theta
\]

\[
= 2 \left[ \frac{1}{2} \cos 2\theta \right]_0^{\pi/2}
\]

\[
= \left[ \cos \pi - \cos 0 \right] = -2
\]

\[
\text{Total area (of both petals)} = 2 \times 2 = 4
\]

6. inner loop of $r = 1 + 2\cos \theta$

\[
\text{Area} = \frac{1}{2} \int_{\theta_1}^{\theta_2} (1 + 2\cos \theta)^2 \, d\theta
\]

\[
= \frac{1}{2} \int_{\theta_1}^{\theta_2} (1 + 4\cos \theta + 4\cos^2 \theta) \, d\theta
\]

\[
= \frac{1}{2} \int_{\theta_1}^{\theta_2} (1 + 4\cos \theta + 2 + 2\cos 2\theta) \, d\theta
\]

\[
= \frac{1}{2} \left[ \theta + 4\sin \theta + \theta + \sin 2\theta \right]_{\theta_1}^{\theta_2}
\]

\[
= \theta + 3\sin \theta
\]

\[
\approx 8.338
\]

7. between the loops of $r = 1 + 2\cos \theta$

\[
\text{Area} = 2 \left[ \int_0^{\pi/3} (1 + 2\cos \theta) \, d\theta - \frac{1}{2} \int_{\pi/3}^{\pi/2} (1 + 2\cos \theta) \, d\theta \right]
\]

\[
= 2 \left[ \frac{1}{2} \sin \theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/3} - \frac{1}{2} \left[ \theta + \sin \theta \right]_{\pi/3}^{\pi/2}
\]

\[
= 2 \left( \frac{1}{2} \sin \frac{\pi}{3} + \frac{1}{2} \sin \frac{2\pi}{3} \right) - \frac{1}{2} \left( \frac{\pi}{2} + \sin \frac{\pi}{3} - \frac{\pi}{2} - \sin \frac{\pi}{3} \right)
\]

\[
= \frac{3\sqrt{3}}{2} - \frac{\pi}{6}
\]

\[
\approx 8.337
\]

8. one loop of $r^2 = 4\cos(2\theta)$

\[
\text{Area} = \frac{1}{2} \int_0^{\pi/4} (4\cos 2\theta) \, d\theta
\]

\[
= 2\left( \frac{1}{4} \sin 2\theta \right)_{\theta_1}^{\pi/4}
\]

\[
= \frac{1}{2} \left( \sin \frac{\pi}{2} - \sin 0 \right)
\]

\[
= \frac{1}{2}
\]
9. inside \( r = 3\cos \theta \) and outside \( r = 2 - \cos \theta \)

\[
\text{Area} = 2 \left[ \frac{1}{2} \int_0^{\pi/3} (3\cos \theta)^2 d\theta - \frac{1}{2} \int_0^{\pi/3} (2 - \cos \theta)^2 d\theta \right]
\]

Symmetry
\[
\int_0^{\pi/3} \left[ 9\cos \theta - (2 - \cos \theta)^2 \right] d\theta
\]

\[\approx 3\sqrt{3}\]
\[\approx 5.196\]

10. common interior of \( r = 4\sin \theta \) and \( r = 2 \)

\[
\text{Area} = 2 \left[ \frac{1}{2} \int_0^{\pi/6} (4\sin \theta)^2 d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/2} (2)^2 d\theta \right]
\]

Symmetry
\[
\int_0^{\pi/6} (4\sin \theta)^2 d\theta + \int_{\pi/6}^{\pi/2} 4 d\theta
\]

\[= \frac{8\pi}{3} - 2\sqrt{3}\]
\[\approx 4.913\]

11. inside \( r = 3\sin \theta \) and outside \( r = 1 + \sin \theta \)

\[
\text{Area} = 2 \left[ \frac{1}{2} \int_0^{\pi/2} (3\sin \theta)^2 d\theta - \left(1 + \sin \theta\right)^2 \right] d\theta
\]

Symmetry
\[
\int_0^{\pi/2} \left[ 9\sin \theta - (1 + \sin \theta)^2 \right] d\theta
\]

\[= \pi\]
\[\approx 3.141\text{ or }3.142\]

12. common interior of \( r = 3\cos \theta \) and \( r = 1 + \cos \theta \)

\[
\text{Area} = 2 \left[ \frac{1}{2} \int_0^{\pi/3} (3\cos \theta)^2 d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} (1 + \cos \theta)^2 d\theta \right]
\]

Symmetry
\[
\int_0^{\pi/3} (3\cos \theta)^2 d\theta + \int_{\pi/3}^{\pi/2} (1 + \cos \theta)^2 d\theta
\]

\[= \frac{5\pi}{4}\]
\[\approx 3.927\text{ or }3.927\]
13. common interior of $r = 4 \sin (2\theta)$ and $r = 2$

$$\text{Find 1 sliver, then multiply by 4 petals}$$

$$\text{Area} = \frac{1}{2} \int_0^{\pi/2} \left( \left(4 \sin (2\theta) \right)^2 \right) d\theta + \frac{1}{2} \int_{\pi/2}^{\pi} \left(2\right)^2 d\theta$$

$$= 4 \int_0^{\pi/2} \sin^2 (2\theta) d\theta + \int_{\pi/2}^{\pi} 4 d\theta$$

$$= 64 \int_0^{\pi/2} \sin^2 (2\theta) d\theta + \int_{\pi/2}^{\pi} 8 d\theta$$

$$= 9.826 \text{ or } 9.827$$

14. inside $r = 2$ and outside $r = 2 - \sin \theta$

$$\text{Area} = 2 \left[ \frac{1}{2} \int_0^{\pi/2} \left( (2)^2 - (2 - \sin \theta)^2 \right) d\theta \right]$$

$$= \int_0^{\pi/2} (4 - (2 - \sin \theta)^2) d\theta$$

$$= 4 - \frac{\pi}{4}$$

$$\approx 3.214 \text{ or } 3.215$$

15. inside $r = 2 + 2 \cos (2\theta)$ and outside $r = 2$

$$\text{Area} = 4 \left[ \frac{1}{2} \int_0^{\pi/4} \left( (2+2 \cos 2\theta)^2 - (2)^2 \right) d\theta \right]$$

$$= 2 \int_0^{\pi/4} \left( (2+2 \cos 2\theta)^2 - 4 \right) d\theta$$

$$= 11.5707$$
Free Response

16. The figure shows the graphs of the line $y = \frac{2}{3}x$ and the curve $C$ given by $y = \sqrt{1 - \frac{x^2}{4}}$. Let $S$ be the region in the first quadrant bounded by the two graphs and the $x$-axis. The line and the curve intersect at point $P$.

\[
\text{Area} = \int_0^1 \left( \frac{2}{3}x - 0 \right) dx + \int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left( \sqrt{1 - \frac{x^2}{4}} - 0 \right) dx
\]

(b) Find a polar equation to represent curve $C$.

\[
\begin{align*}
\sin \theta &= \frac{\sqrt{1 - x^2}}{2} \\
\cos \theta &= \frac{x}{2} \\
\tan \theta &= \frac{\sqrt{1 - x^2}}{x} \\
\theta &= \arccos \left( \frac{x}{2} \right)
\end{align*}
\]

(d) Use the polar equation found in (c) to set up and evaluate an integral expression with respect to the polar angle $\theta$ that gives the area of $S$.

\[
\text{Area} = \frac{1}{2} \int_0^{\arccos \left( \frac{1}{2} \right)} \frac{4}{\sin^2 \theta + \cos^2 \theta} d\theta
\]

$\theta = \arccos \left( \frac{1}{2} \right)$

$\theta = 0, \frac{\pi}{3}$

$\text{Area} = 0.927$
17. A curve is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \cos(3\theta)$ for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$, where $r$ is measured in meters and $\theta$ is measured in radians.

(a) Find the area bounded by the curve and the y-axis.

\[
\text{Area} = \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\theta + \cos(3\theta)) \, d\theta = 19.674 \text{ or } 19.675
\]

(b) Find the angle $\theta$ that corresponds to the point on the curve with y-coordinate $-1$.

\[
y = -1 \\
(\theta + \cos(3\theta))\sin\theta = -1 \\
(\theta + \cos(3\theta))\sin\theta + 1 = 0 \\
\theta = 3.484 \text{ or } 3.985
\]

(c) For what values of $\theta$, $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$ is \( \frac{dr}{d\theta} \) positive? What does this say about $r$?

\[
r = \theta + \cos(3\theta) \\
\frac{dr}{d\theta} = 1 - 3\sin(3\theta) > 0 \text{ at } 2.08 \text{ (radian mode)} \\
r \in (2.207, 2.08) \cup (3.028, 4.302)
\]

On these intervals, the graph of $r(\theta)$ is moving away from the pole/origin.

(d) Find the value of $\theta$ on the interval $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$ that corresponds to the point on the curve with the greatest distance from the origin. What is this greatest distance? Justify your answer.

\[
\text{Maximize } r \\
\frac{dr}{d\theta} = 0 \\
r = 2.207 = A \text{ (smallest)} \\
r = 3.028 = B \\
r = 4.302 = C
\]

\[
\begin{align*}
\text{at } D &= 4.302 \text{ radians, at this angle,} \\
\text{the graph is furthest from pole/origin} \\
\text{the graph is 5.244 units from the pole/origin.}
\end{align*}
\]
18. A region $R$ in the $xy$-plane is bounded below by the $x$-axis and above by the polar curve defined by \( r = \frac{4}{1 + \sin \theta} \) for $0 \leq \theta \leq \pi$.

(a) Find the area of $R$ by evaluating an integral in polar coordinates.

\[
\text{Area} = \frac{1}{2} \int_{0}^{\pi} \left( \frac{4}{1 + \sin \theta} \right)^2 \, d\theta
\]

\[
= \frac{1}{2} \int_{0}^{\pi} \frac{16}{(1 + \sin \theta)^2} \, d\theta
\]

\[
= 10.666 \text{ or } 10.667 \text{ or } \frac{20}{3}
\]

(b) The curve resembles an arch of the parabola $8y = 16 - x^2$. Convert the polar equation to rectangular coordinates, and prove that the curves are the same.

\[
r = \frac{4}{1 + \sin \theta}
\]

\[
\Rightarrow \quad y = \frac{4r}{r + y}
\]

\[
r + y = 4
\]

\[
r = \frac{4 - y}{r + y}
\]

\[
\sqrt{x^2 + y^2} = \frac{4 - y}{r + y}
\]

\[
x^2 + y^2 = 16 - 8y + y^2
\]

\[
\frac{8y}{2} = 16 - x^2
\]

\[
y = 2 - \frac{1}{8}x^2
\]

(c) Set up an integral in rectangular coordinates that gives the area of $R$.

\[
\text{using symmetry} \quad \text{Area} = 2 \int_{0}^{4} \left( 2 - \frac{1}{8}x^2 \right) \, dx
\]

\[
\text{or without symmetry} \quad \text{Area} = \int_{-4}^{4} \left( 2 - \frac{1}{8}x^2 \right) \, dx
\]

\[
x = 2 - \frac{1}{8}x^2 = 0
\]

\[
z = \frac{1}{8}x^2 = 0
\]

\[
l = x^2
\]

\[
x = \pm 4
\]
Multiple Choice

19. Which of the following is equal to the area of the region inside the polar curve \( r = 2 \cos \theta \) and outside the polar curve \( r = \cos \theta \)?

\[ \begin{align*}
(A) & \quad \frac{3}{2} \int_0^{\pi} \cos^2 \theta \, d\theta \\
(B) & \quad \frac{3}{2} \int_0^{\pi} \sin^2 \theta \, d\theta \\
(C) & \quad \frac{3}{2} \int_0^{\pi} \cos \theta \, d\theta \\
(D) & \quad \frac{3}{2} \int_0^{\pi} \cos \theta \, d\theta \\
(E) & \quad \frac{3}{2} \int_0^{\pi} \cos \theta \, d\theta
\end{align*} \]

Area = \( \frac{1}{2} \int_0^{\pi} (2 \cos^2 \theta - \cos^2 \theta) \, d\theta = \frac{1}{2} \int_0^{\pi} (\cos^2 \theta - \cos^2 \theta) \, d\theta = \frac{3}{2} \int_0^{\pi} \cos^2 \theta \, d\theta \) (Not shown!)

20. The area of the region enclosed by the polar graph of \( r = \sqrt{3 + \cos \theta} \) is given by which integral?

\[ \begin{align*}
(A) & \quad \int_0^{2\pi} \sqrt{3 + \cos \theta} \, d\theta \\
(B) & \quad \int_0^{2\pi} \sqrt{3 + \cos \theta} \, d\theta \\
(C) & \quad \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (3 + \cos \theta) \, d\theta \\
(D) & \quad \int_0^{\pi} (3 + \cos \theta) \, d\theta \\
(E) & \quad \int_0^{\pi} \sqrt{3 + \cos \theta} \, d\theta
\end{align*} \]

Area = \( \frac{1}{2} \int_0^{2\pi} (\sqrt{3 + \cos \theta})^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (3 + \cos \theta) \, d\theta \) (Not shown!)

21. The area enclosed by one petal of the 3-petaled rose curve \( r = 4 \cos (3\theta) \) is given by which integral?

\[ \begin{align*}
(A) & \quad 16 \int_{-\pi/3}^{\pi/3} \cos (3\theta) \, d\theta \\
(B) & \quad 8 \int_{-\pi/6}^{\pi/6} \cos (3\theta) \, d\theta \\
(C) & \quad 8 \int_{-\pi/3}^{\pi/3} \cos^2 (3\theta) \, d\theta \\
(D) & \quad 16 \int_{-\pi/6}^{\pi/6} \cos (3\theta) \, d\theta \\
(E) & \quad 8 \int_{-\pi/6}^{\pi/6} \cos^2 (3\theta) \, d\theta
\end{align*} \]

Area = \( 2 \left[ \frac{1}{2} \int_0^{\pi/6} (4 \cos^2 (3\theta)) \, d\theta \right] = 8 \int_0^{\pi/6} \cos^2 (3\theta) \, d\theta \) (Not shown!)