① (a) $\int_5^{\infty} \frac{1}{(x-2)^2}\,dx$

* Improper since the upper limit of integration is infinite.

(b) $\int_1^{5} \frac{1}{(x-2)^2}\,dx$

* Improper since the VA at $x=2$ is on the interior of the interval of integration.

(c) $\int_2^{5} \frac{1}{(x-2)^2}\,dx$

* Improper since the VA at $x=2$ is the lower limit of integration

(d) $\int_3^{5} \frac{1}{(x-2)^2}\,dx$

* Not Improper since interval is to right of the VA at $x=2$.

③ $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges $\forall p>1$.

$\int_a^{\infty} \frac{1}{x^p}\,dx$ <u>converges</u> $\forall p>1$ and $\forall a>0$, <u>diverges</u> for $p \le 1$ or $a \le 0$

* if $a=1$, $p>1$, then $\int_a^{\infty} \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$

$= \lim_{b\to\infty} \int_a^b x^{-p}\,dx$

$= \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_a^b$

$= \lim_{b\to\infty} \left(\frac{1}{-p+1}\left(b^{-p+1} - a^{-p+1}\right)\right)$

* for $p>1$, $-p+1<0$ so

$\lim_{b\to\infty} b^{-p+1} = 0$ converges to

$= \frac{-a^{-p+1}}{-p+1} = \boxed{\frac{a^{1-p}}{p-1}}$

Improper [illegible] of Improper Integral problems.

② (a) if $\int_a^{\infty} f(x)\,dx = K$, $0<g(x)\le f(x)$, then $\int_a^{\infty} g(x)\,dx$ <u>Converges</u> to a value $\le K$

(b) if $\int_a^{\infty} f(x)\,dx = K$, $0<f(x)<g(x)$, then $\int_a^{\infty} g(x)$ <u>May</u> or <u>May not</u> converge/inconclusive

(c) if $\int_a^{\infty} f(x)\,dx$ diverges, $0<f(x)\le g(x)$, then $\int_a^{\infty} g(x)$ diverges too!

(d) if $\int_a^{\infty} f(x)\,dx$ diverges, $0<g(x)<f(x)$, then $\int_a^{\infty} g(x)$ <u>May</u> or <u>May not</u> diverge/inconclusive

4 (a) $\int_2^{\infty} \frac{x^5}{x^6-1}\,dx$

compare with $\int_2^{\infty} \frac{1}{x}\,dx$, a divergent integral

$\frac{x^5}{x^6-1} > \frac{1}{x}$

since $x^6 > x^6 - 1$ for $x>2$

So $\int_2^{\infty} \frac{x^5}{x^6-1}\,dx$ diverges too!

(b) $\int_2^{\infty} \frac{x^3+1}{(x^4+4x+1)^2}\,dx$

$\approx \int_2^{\infty} \frac{x^3+1}{x^8+\cdots}\,dx$

Compare to $\int_2^{\infty} \frac{1}{x^5}\,dx$ a convergent integral.

$\frac{x^3+1}{x^8+\cdots} < \frac{1}{x^5}$

$x^8+x^5 < x^8+\cdots$ for some $x>2$

So $\int_2^{\infty} \frac{x^3+1}{(x^4+4x+1)^2}$ converges too!

(c) $\int_1^{\infty} \frac{1}{(x+5)^5}\,dx$

compare with $\int_1^{\infty} \frac{1}{x^5}\,dx$ a convergent integral

$\frac{1}{(x+5)^5} < \frac{1}{x^5}$

$x^5 < (x+5)^5$ for $x>1$

So $\int_1^{\infty} \frac{1}{(x+5)^5}\,dx$ converges too!

(d) $\int_4^{\infty} \frac{3+\sin x}{x}\,dx$

compare with $\int_4^{\infty} \frac{2}{x}\,dx$ & $\int_4^{\infty} \frac{4}{x}$ two divergent integrals

since $2 \le 3+\sin x \le 4$

$\int_4^{\infty} \frac{3+\sin x}{x}\,dx$ diverges too!

5 $\int_0^{\infty} x^2 e^{-x^3}\,dx$

$= \lim_{b\to\infty} -\frac{1}{3}e^{-x^3}\Big|_0^b$

$= -\frac{1}{3}\lim_{b\to\infty}\left(e^{-b^3}-e^0\right)$

$= -\frac{1}{3}(0-1)$

$= \boxed{\frac{1}{3}}\ \boxed{C}$

6 $\int_1^{\infty} \frac{1}{x^{1.01}}\,dx$

$= \frac{1}{1.01-1}$

$= \frac{1}{1/100}$

$= \boxed{100}\ \boxed{C}$

7 $\int_0^1 \frac{1}{x^{1/2}}\,dx$

$\lim_{b\to 0^+} \int_b^1 x^{-1/2}\,dx$

$\lim_{b\to 0^+} 2x^{1/2}\Big|_b^1$

$\lim_{b\to 0^+} 2\left[1-b^{1/2}\right]$

$= \boxed{2}\ \boxed{B}$

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(8) $\int_0^1 \frac{1}{x-1}\,dx$

$= \lim_{b\to 1^-} \ln|x-1| \Big|_0^b$

$= \lim_{b\to 1^-} \ln|b-1| - \ln|-1|$

$= \lim_{b\to 1^-} \ln|b-1| = -\infty$

$\boxed{E}$ Diverges

(9)

$\int_0^\infty \frac{1}{x^2+1}\,dx$

$\lim_{b\to\infty} \arctan x \Big|_0^b$

$\lim_{b\to\infty} (\arctan b - \arctan 0)$

$\lim_{b\to\infty} \arctan b$

$\boxed{\frac{\pi}{2}}$ $\boxed{C}$

(10) $\int_0^2 f(x)\,dx,\ f(x) = \begin{cases} x^{-1/2}, & x \le 1 \\ x, & 1 < x \le 2 \end{cases}$

$\int_0^1 x^{-1/2}\,dx + \int_1^2 x\,dx$

$= \lim_{b\to 0^+} 2x^{1/2}\Big|_b^1 + \frac{1}{2}x^2\Big|_1^2$

$2\left[\lim_{b\to 0^+} (1 - b^{1/2})\right] + \frac{1}{2}(2^2 - 1^2)$

$2 + 2 - \frac{1}{2} = \boxed{\frac{7}{2}}$ $\boxed{C}$

(11) $\int_2^\infty \frac{x}{\sqrt[3]{x^2-2}}\,dx$

$\lim_{b\to\infty} \int_2^b x(x^2-2)^{-1/3}\,dx$

$\lim_{b\to\infty} \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)(x^2-2)^{2/3}\Big|_2^b$

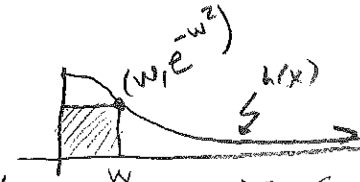
corr rule

$\frac{3}{4} \lim_{b\to\infty} \left[(b^2-2)^{2/3} - 2^{2/3}\right]$

$\frac{3}{4}\left(\infty - 2^{2/3}\right) = \infty$

$\boxed{E}$ $\boxed{\text{Diverges}}$

(12) $h(x) = e^{-x^2},\ 0 \le x < \infty$



(a) $\circlearrowleft$ y-axis, vertical slice/vertical axis, so ParaSHELL

$\text{Volume} = 2\pi \int_0^\infty x(e^{-x^2} - 0)\,dx$

$= 2\pi \lim_{b\to\infty} \left[\left(-\frac{1}{2}\right)e^{-x^2}\right]\Big|_0^b$

$= \frac{2\pi}{-2} \lim_{b\to\infty} \left(e^{-b^2} - e^0\right)$

$= -\pi(-1) = \boxed{\pi}$

(b) $A(w) = we^{-w^2}$ (width × height), $w \in (0, \infty)$

$A'(w) = e^{-w^2} + w(-2w)e^{-w^2} = 0$

$e^{-w^2}(1 - 2w^2) = 0$

$e^{-w^2} = 0$ or $1 - 2w^2 = 0$

(Never) $w = -\cancel{\frac{1}{\sqrt{2}}}$ or $\boxed{\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = w}$ critical value

$A''(w) = -2we^{-w^2} + e^{-w^2}(-4w)$

$= e^{-w^2}(-2w - 4w)$

$A''(w) = -6we^{-w^2} < 0\ \forall w \in (0, \infty)$, so $A(w)$ is concave down $\forall w \in (0, \infty)$ and $w = \frac{\sqrt{2}}{2}$ is an absolute Max.

* find inflection value of $h(x)$

$h(x) = e^{-x^2}$

$h'(x) = -2xe^{-x^2}$

$h''(x) = -2e^{-x^2} - 2x(-2x)e^{-x^2} = 0$

$e^{-x^2}(-2 + 4x^2) = 0$

$e^{-x^2} = 0$ or $4x^2 - 2 = 0$

(Never) $x = -\cancel{\frac{1}{\sqrt{2}}}$ or $\frac{1}{\sqrt{2}}$

Since $h''(x) < 0$ for $x \in (0, \frac{1}{\sqrt{2}})$ and $h''(x) > 0$ for $x > \frac{1}{\sqrt{2}}$, $h(x)$ has an inflection point at $x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

(13) $f'(x) = -3x f(x) \ \forall x \in \mathbb{R}$, $f(1) = 4$, $\lim_{x\to\infty} f(x) = 0$

(a) $\int_1^{\infty} -3x f(x)\,dx$

$= \int_1^{\infty} f'(x)\,dx$

$= \lim_{b\to\infty} f(x)\Big|_1^b$ ✓

$= \lim_{b\to\infty} \left(f(b) - f(1)\right)$

$= 0 - 4$

$= \boxed{-4}$ ✓

(b) Euler's Method. $x = 1$, $\Delta x = 0.5$, $f(2) \approx ?$

✓

x	y	$f'(x,y) = m$	$\Delta y = m\Delta x$	y_{new}
1	4	$-3f(1) = -12$	-6	-2
1.5	-2	$-3(1.5)f(1.5) = -4.5(-2) = 9$	4.5	2.5
2	2.5			

So $\boxed{f(2) \approx 2.5}$ ✓

(c) $\frac{dy}{dx} = -3xy$, $f(1) = 4 = y(1)$

✓ $\int \frac{1}{y}\,dy = \int -3x\,dx$

✓ $\ln|y| = -\frac{3}{2}x^2 + C$ ✓

$|y| = e^{-3/2x^2 + C}$

$y = Ce^{-\frac{3}{2}x^2}$ gen soln.

when $x=1, y=4$:

$4 = Ce^{-3/2}$

$\boxed{C = 4e^{3/2}}$ ✓

So $y = 4e^{3/2} \cdot e^{-3/2x^2}$ ✓ particular soln

or

$y = 4e^{\left(\frac{3}{2} - \frac{3}{2}x^2\right)}$

(14) $f(x) = \frac{1}{x}$, $g(x) = \frac{4x}{1+4x^2}$ $\forall x > 0$

(a) $g'(x) = \frac{(1+4x^2)(4) - (4x)(8x)}{(1+4x^2)^2} = \frac{4(1-4x^2)}{(1+4x^2)^2}$

$g'(x) = 0$ when $1 - 4x^2 = 0$

$x = \cancel{-\frac{1}{2}}$ or $\boxed{x = \frac{1}{2}}$ critical value

x	$x=0$	$1/4$	$x=\frac{1}{2}$	1
$g'(x)$		$+$		$-$

. MAX VALUE is $g(\frac{1}{2}) = 1$

* $g(x)$ has an Absolute Max of 1 at $x = \frac{1}{2}$ since $g'(x) > 0$ for all $0 < x < \frac{1}{2}$ and $g'(x) < 0$ for all $x > \frac{1}{2}$

* $g(x)$ has No Absolute minimum for $x > 0$ (No endpoints, $x \in (0, \infty)$)

(b) From verbal description, for $x > 1$ $f(x) > g(x) \geq 0$

$$\text{Area} = \int_1^\infty (f(x) - g(x))\,dx$$

$$= \lim_{b\to\infty} \int_1^b \left(\frac{1}{x} - \frac{4x}{1+4x^2}\right) dx$$

$$= \lim_{b\to\infty} \left[\ln|x| - \frac{1}{2}\ln|1+4x^2|\right]\Big|_1^b$$

$$= \lim_{b\to\infty} \left[\left(\ln b - \frac{1}{2}\ln(1+4b^2)\right) - \left(\cancelto{0}{\ln 1} - \frac{1}{2}\ln(5)\right)\right]$$

$$= \lim_{b\to\infty} \left(\ln b - \ln\sqrt{1+4b^2} + \ln\sqrt{5}\right)$$ *condensing

$$= \lim_{b\to\infty} \ln\left[\frac{b\sqrt{5}}{\sqrt{1+4b^2}}\right]$$

* by growth analysis, the numerator is $\sqrt{5}\cdot b$, the denominator is effectively $\sqrt{4b^2} = 2b$, so the limit approaches $\ln\frac{\sqrt{5}}{2}$

$$= \ln\left[\lim_{b\to\infty} \frac{b\sqrt{5}}{\sqrt{1+4b^2}}\right]$$

$$= \boxed{\ln\frac{\sqrt{5}}{2} \text{ or } \ln\sqrt{\frac{5}{4}} \text{ or } \frac{1}{2}\ln\frac{5}{4}}$$

or $\lim_{b\to\infty} \ln\left[\frac{\sqrt{5b^2}}{\sqrt{4b^2+1}}\right]$

$$= \lim_{b\to\infty} \ln\left(\frac{5b^2}{4b^2+1}\right)^{1/2}$$

$$= \frac{1}{2}\ln\left(\lim_{b\to\infty} \frac{5b^2}{4b^2+1}\right)$$

$$= \frac{1}{2}\ln\frac{5}{4}$$