Worksheet 6.6—Improper Integrals
Show all work. No calculator unless explicitly stated.

Short Answer
1. Classify each of the integrals as proper or improper integrals. Give a clear reason for each.

(a) \( \int_{5}^{\infty} \frac{dx}{(x-2)^2} \)  
(b) \( \int_{1}^{5} \frac{dx}{(x-2)^2} \)  
(c) \( \int_{2}^{5} \frac{dx}{(x-2)^2} \)  
(d) \( \int_{3}^{5} \frac{dx}{(x-2)^2} \)

2. Answer the following.
   
   (a) If \( \int_{a}^{\infty} f(x)\,dx = K \) and \( 0 < g(x) \leq f(x) \), what can we say about \( \int_{a}^{\infty} g(x)\,dx \)?

   (b) If \( \int_{a}^{\infty} f(x)\,dx = K \) and \( 0 < f(x) < g(x) \), what can we say about \( \int_{a}^{\infty} g(x)\,dx \)?

   (c) If \( \int_{a}^{\infty} f(x)\,dx \) diverges and \( 0 < f(x) \leq g(x) \), what can we say about \( \int_{a}^{\infty} g(x)\,dx \)?

   (d) If \( \int_{a}^{\infty} f(x)\,dx \) diverges and \( 0 < g(x) < f(x) \), what can we say about \( \int_{a}^{\infty} g(x)\,dx \)?
3. If \( \int_{1}^{\infty} \frac{1}{x^p} \, dx \) converges for \( p > 1 \), what can be said in general about improper integrals of the form
\[
\int_{a}^{\infty} \frac{1}{x^p} \, dx
\]
For what values of \( a \) does the function diverge? Converge? To what?

4. Determine if the improper integral converges or diverges by finding a function to compare it to. Justify by showing the inequality and discussing the convergence/divergence of the function to which you compare.

(a) \( \int_{2}^{\infty} \frac{x^5}{x^6 - 1} \, dx \)  
(b) \( \int_{2}^{\infty} \frac{x^3 + 1}{(x^4 + 4x + 1)^2} \, dx \)  
(c) \( \int_{1}^{\infty} \frac{dx}{(x + 5)^5} \)  
(d) \( \int_{4}^{\infty} \frac{3 + \sin x}{x} \, dx \)

Multiple Choice

5. \( \int_{0}^{\infty} x^2 e^{-x^3} \, dx = \)

(A) \( -\frac{1}{3} \)  
(B) 0  
(C) \( \frac{1}{3} \)  
(D) 1  
(E) Diverges
6. Which of the following gives the value of the integral \( \int_{1}^{\infty} \frac{dx}{x^{1.01}} \)?

(A) 1  (B) 10  (C) 100  (D) 1000  (E) Diverges

7. Which of the following gives the value of the integral \( \int_{0}^{1} \frac{dx}{x^{0.5}} \)?

(A) 1  (B) 2  (C) 3  (D) 4  (E) Diverges

8. Which of the following gives the value of the integral \( \int_{0}^{1} \frac{dx}{x - 1} \)?

(A) \(-1\)  (B) \(-1/2\)  (C) 0  (D) 1  (E) Diverges
9. Which of the following gives the value of the area under the curve \( y = \frac{1}{x^2 + 1} \) in the first quadrant?

(A) \( \frac{\pi}{4} \)  \hspace{1cm}  (B) 1  \hspace{1cm}  (C) \( \frac{\pi}{2} \)  \hspace{1cm}  (D) \( \pi \)  \hspace{1cm}  (E) Diverges

10. Determine if \( \int_{0}^{2} f(x) \, dx \) is convergent or divergent when \( f(x) = \begin{cases} 
    x^{-1/2}, & \text{if } x \leq 1 \\
    x, & \text{if } 1 < x \leq 2
  \end{cases} \), and if it is convergent, find its value.

(A) 1/2  \hspace{1cm}  (B) 5/2  \hspace{1cm}  (C) 7/2  \hspace{1cm}  (D) 4  \hspace{1cm}  (E) Diverges

11. \( \int_{2}^{\infty} \frac{x}{3\sqrt{x^2 - 2}} \, dx = \)

(A) \( \frac{3 \cdot 2^{2/3}}{4} \)  \hspace{1cm}  (B) \( 2^{2/3} \)  \hspace{1cm}  (C) \( -\frac{3 \cdot 2^{2/3}}{4} \)  \hspace{1cm}  (D) \( -\frac{3 \cdot 2^{2/3}}{2} \)  \hspace{1cm}  (E) Diverges
Free Response

12. (AP 1996-1) Consider the graph of the function $h$ given by $h(x) = e^{-x^2}$ for $0 \leq x < \infty$.

(a) Let $R$ be the unbounded region in the first quadrant below the graph of $h$. Find the volume of the solid generated when $R$ is revolved about the $y$-axis.

(b) Let $A(w)$ be the area of the shaded rectangle shown in the figure. Show that $A(w)$ has its maximum value when $w$ is the $x$-coordinate of the point of inflection of the graph of $h$. 
13. (AP 2001-5) Let \( f \) be the function satisfying \( f'(x) = -3xf(x) \), for all real numbers \( x \), with \( f(1) = 4 \) and \( \lim_{x \to \infty} f(x) = 0 \).

(a) Evaluate \( \int_{1}^{\infty} -3xf(x) \, dx \). Show the work that leads to your answer.

(b) Use Euler’s method, starting at \( x = 1 \) with a step size of 0.5, to approximate \( f(2) \).

(c) Write an expression for \( y = f(x) \) by solving the differential equation \( \frac{dy}{dx} = -3xy \) with the initial condition \( f(1) = 4 \).
14. (AP 2010B-5) Let \( f \) and \( g \) be the functions defined by \( f(x) = \frac{1}{x} \) and \( g(x) = \frac{4x}{1 + 4x^2} \), for all \( x > 0 \).

(a) Find the absolute maximum value of \( g \) on the open interval \((0, \infty)\) if the maximum exists. Find the absolute minimum value of \( g \) on the open interval \((0, \infty)\) if the minimum exists. Justify your answers.

(b) Find the area of the unbounded region in the first quadrant to the right of the vertical line \( x = 1 \), below the graph of \( f \), and above the graph of \( g \).