Name____ Date_______________________ Period______

Worksheet 9.5—Lagrange Error Bound
Show all work. Calculator permitted except unless specifically stated.

Free Response & Short Answer

1. (a) Find the fourth-degree Taylor polynomial for $\cos x$ about $x = 0$. Then use your polynomial to approximate the value of $\cos 0.8$, and use Taylor's Theorem to determine the accuracy of the approximation. Give three decimal places.

(b) Find the interval $[a, b]$ such that $a \le \cos 0.8 \le b$.

(c) Could $\cos 0.8$ equal 0.695? Show why or why not.

2. (a) Write a fourth-degree Maclaurin polynomial for $f(x)=e^x$. Then use your polynomial to approximate e^{-1}, and find a Lagrange error bound for the maximum error when $|x|\le 1$. Give three decimal places.

(b) Find an interval $[a,b]$ such that $a\le e^{-1}\le b$.

3. Let f be a function that has derivatives of all orders for all real numbers x. Assume that $f(5)=6$, $f'(5)=8$, $f''(5)=30$, $f'''(5)=48$, and $\left|f^{(4)}(x)\right| \le 75$ for all x in the interval $[5,5.2]$.

(a) Find the third-degree Taylor polynomial about $x=5$ for $f(x)$.

(b) Use your answer to part (a) to estimate the value of $f(5.2)$. What is the maximum possible error in making this estimate? Give three decimal places.

(c) Find an interval $[a,b]$ such that $a \le f(5.2) \le b$. Give three decimal places.

(d) Could $f(5.2)$ equal 8.254? Show why or why not.

Review (Problems 4 - 7):

4. Find the first four nonzero terms of the power series for $f(x) = \sin x$ centered at $x = \dfrac{3\pi}{4}$.

5. Find the first four nonzero terms and the general term for the Maclaurin series for

(a) $f(x) = x\cos\left(x^3\right)$ (b) $g(x) = \dfrac{1}{1+x^2}$

6. Find the radius and interval of convergence for

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3^n n^2}$ (b) $\displaystyle\sum_{n=0}^{\infty} (2n)!(x-5)^n$

7. Use the Maclaurin series for $\cos x$ to find $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x}$.

8. The Taylor series about $x = 3$ for a certain function *f* converges to $f(x)$ for all *x* in the interval of convergence. The *n*th derivative of *f* at $x = 3$ is given by

$$f^{(n)}(3) = \frac{(-1)^n \, n!}{5^n (n+3)} \text{ and } f(3) = \frac{1}{3}$$

(a) Write the fourth-degree Taylor polynomial for *f* about $x = 3$.

(b) Find the radius of convergence of the Taylor series for *f* about $x = 3$.

(c) Show that the third-degree Taylor polynomial approximates $f(4)$ with an error less than $\dfrac{1}{4000}$.

9. Let f be a function that has derivatives of all orders on the interval $(-1,1)$. Assume $f(0)=1$, $f'(0)=\frac{1}{2}$, $f''(0)=-\frac{1}{4}$, $f'''(0)=\frac{3}{8}$, and $\left|f^{(4)}(x)\right|\le 6$ for all x in the interval $(-1,1)$.

(a) Find the third-degree Taylor polynomial about $x=0$ for the function f.

(b) Use your answer to part (a) to estimate the value of $f(0.5)$.

(d) What is the maximum possible error for the approximation made in part (b)?

10. Let *f* be the function defined by $f(x)=\sqrt{x}$.

(a) Find the second-degree Taylor polynomial about $x=4$ for the function *f*.

(b) Use your answer to part (a) to estimate the value of $f(4.2)$.

(c) Find a bound on the error for the approximation in part (b).

11. Let $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{2^n}$ for all x for which the series converges.

(a) Find the interval of convergence of this series.

(b) Use the first three terms of this series to approximate $f\left(-\frac{1}{2}\right)$.

(c) Estimate the error involved in the approximation in part (b). Show your reasoning.

12. Let f be the function given by $f(x)=\cos\left(3x+\frac{\pi}{6}\right)$ and let $P(x)$ be the fourth-degree Taylor polynomial for f about $x=0$.

(a) Find $P(x)$.

(b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right)-P\left(\frac{1}{6}\right)\right|<\frac{1}{3000}$.

13. (Review) Use series to find an estimate for $I = \int_0^1 e^{-x^2}\,dx$ that is within 0.001 of the actual value. Justify.

14. The Taylor series about $x = 5$ for a certain function *f* converges to $f(x)$ for all *x* in the interval of convergence. The *n*th derivative of *f* at $x = 5$ is given by

$$f^{(n)}(5) = \frac{(-1)^n\, n!}{2^n\,(n+2)} \text{ and } f(5) = \frac{1}{2}.$$

Show that the sixth-degree Taylor polynomial for *f* about $x = 5$ approximates $f(6)$ with an error less than $\frac{1}{1000}$.

Multiple Choice

15. Suppose a function f is approximated with a fourth-degree Taylor polynomial about $x = 1$. If the maximum value of the fifth derivative between $x = 1$ and $x = 3$ is 0.01, that is, $\left| f^{(5)}(x) \right| < 0.01$, then the maximum error incurred using this approximation to compute $f(3)$ is

(A) 0.054 (B) 0.0054 (C) 0.26667 (D) 0.02667 (E) 0.00267

16. What are all the values of x for which the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges?

(A) $-1 \le x \le 1$ (B) $-1 < x < 1$ (C) $-1 < x \le 1$ (D) $-1 \le x < 1$ (E) All real x

17. The coefficient of x^6 in the Taylor series expansion about $x = 0$ for $f(x) = \sin\left(x^2\right)$ is

(A) $-\frac{1}{6}$ (B) 0 (C) $\frac{1}{120}$ (D) $\frac{1}{6}$ (E) 1

18. The maximum error incurred by approximating the sum of the series $1 - \frac{1}{2!} + \frac{2}{3!} - \frac{3}{4!} + \frac{4}{5!} - \cdots$ by the sum of the first six terms is

(A) 0.001190 (B) 0.006944 (C) 0.33333 (D) 0.125000 (E) None of these

19. If f is a function such that $f'(x) = \sin\left(x^2\right)$, then the coefficient of x^7 in the Taylor series for $f(x)$ about $x = 0$ is

(A) $\frac{1}{7!}$ (B) $\frac{1}{7}$ (C) 0 (D) $-\frac{1}{42}$ (E) $-\frac{1}{7!}$

20. Now that you have finished the last question of the last “new concept” worksheet of your high school career, how do you feel? (Show your work)

(A) Relieved (B) Very Sad (C) Euphoric (D) Tired (E) All of these