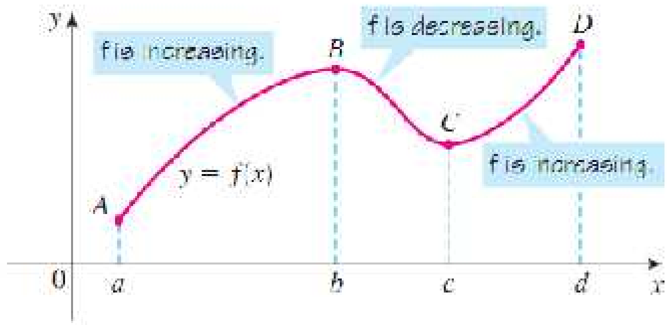
Chapter 2.3: Other Properties of Functions

If the graph of a function is doing anything on an open interval, it must be doing one of three things:

1. Increasing
2. Decreasing
3. Hanging out doing nothing (constant)

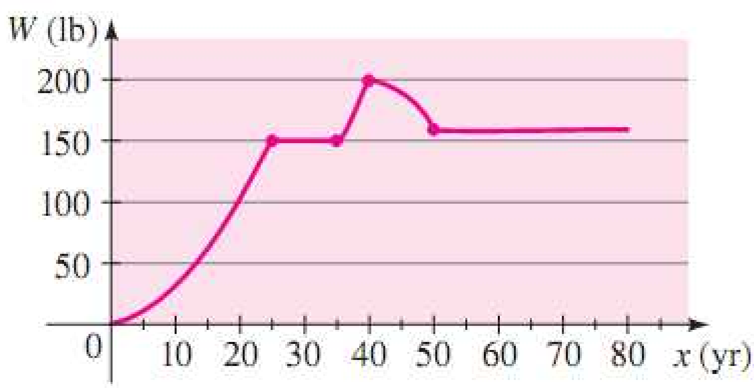
When we read a graph for increasing/decreasing/constant behavior, we must read it like we read the words on a page: from left to right. As we look at increasing *x*-values on an open interval, we are interested in what the *y*-values are doing.



In the graph above, the graph is increasing on $(a,b)\cup(c,d)$ and decreasing on (b,c).

- To avoid fighting over endpoints where functions change increasing/decreasing behavior, we will concern ourselves will addressing open intervals.
- We separate disjoint intervals using the union symbol, $\cup$, a mathy word for "or."

Example 1:
Grandpa George is 80 years old and reminiscing about how his weight has fluctuated over his years. He envisions the following piece-wise graph in his head:



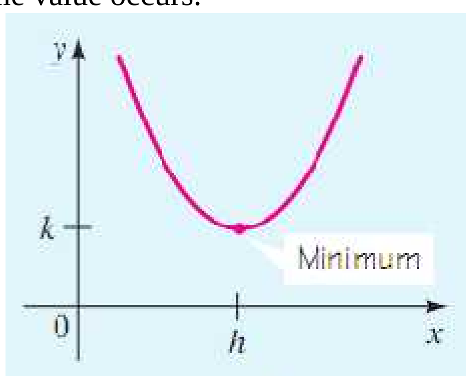
Although Grandpa is prodigious at envisioning such graphs, his knowledge of increasing/decreasing/constant intervals escapes him in his old age. Help him out by defining the open intervals on which his weight (a) increased (b) decreased and (c) remained constant.

Example 2:
For each function, without a calculator, list the open intervals on which the function is either increasing, decreasing, or constant. Do it without a calculator. A sketch might help.

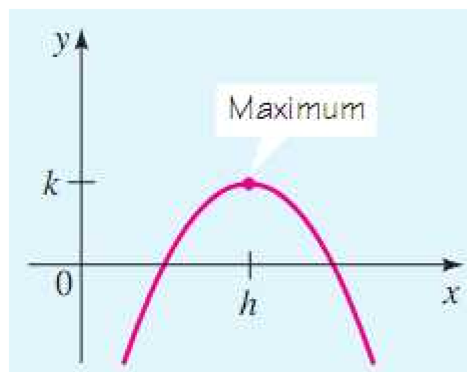
(a) $f(x) = 3x^3$ (b) $g(x) = -5(x-4)^2$ (c) $h(x) = \frac{5}{x+3}$

Functions that change increasing/decreasing behavior turn around vertically. When this happens, we get a point that is called, collectively, as a **Local (or Relative) Extremum.** They can be **Local Maximums** or **Local Minimums**. For now, we can say that a Local Max will occur on a "hill," and a Local Min will occur in a "valley."

The y-value at such a point is the actual relative extreme value, while the x-value is the location at which the extreme value occurs.



Local Min of $y = k$ at $x = h$



Local Min of $y = k$ at $x = h$

For parabolas, the relative extremum is the coordinate of the vertex and can be found algebraically. For most other functions, these extreme values require the use of calculus or a graphing calculator.

Example 3:
Find the coordinates of all relative extrema for the following functions using a graphing calculator.

(a) $f(x) = x^4 - 7x^2 + 6x$ (b) $U(x) = 3(x-4)^{2/3} - 2$ (c) $g(x) = x\sqrt{x - x^2}$

Functions like the ones in the previous example have graphs that are curvy, meaning they increase or decrease at different rates. We are all familiar with speed. For instance, if you drive a total distance of 130 miles in 2 hours, your average speed, or **average rate of change**, would be

$$\text{Avg speed} = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{130 \text{ mi}}{2 \text{ hr}} = 65 \text{ m/h}$$

This doesn't mean, necessarily, that you had to be traveling at 65 mph the entire time, for you likely went slower at times, and therefore, faster than 65 mph at other times.

We can do the same calculation for functions over *x* intervals, talking about how "fast," on average, the *y*-values change on the interval. In the above example, if we let the function $d(t)$ represent the distance you traveled, in miles, and let *t* represent the time spent on your trip, in hours, the calculation looks like this:

$$\text{Avg rate of change of position} = \frac{d(2)-d(0)}{2-0} = \frac{130 \text{ mi}}{2 \text{ hr}} = 65 \text{ m/h}$$

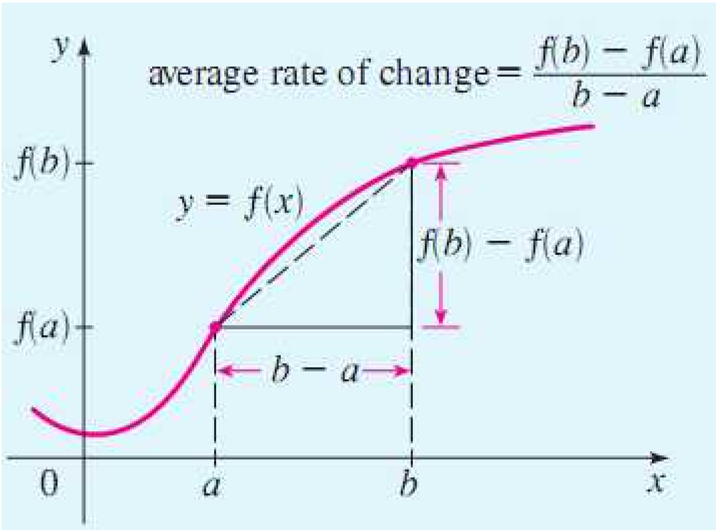
Average Rate of Change on a Closed Interval

The **average rate of change** of the function $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{avg rt o' change} = \frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$$

The above is called a difference quotient, and is the same difference quotient you learned in Algebra I when you were learning how to calculate slope.

There is a graphical interpretation of this. The average rate of change is the slope of the **secant line** between $x = a$ and $x = b$ on the graph of *f*, that is, the line passing through the points $(a, f(a))$ and $(b, f(b))$.



Example 4:

Sketch the function (without a calculator) $f(x)=(x-4)^2$, then find the average rate of change on the following closed intervals:

(a) $x \in [2,4]$ (b) $5 \le x \le 8$ (c) $x \in [-1,9]$

.

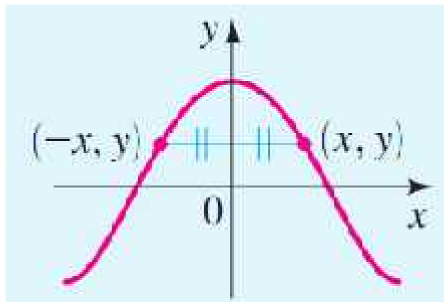
Example 5:

When a Precalculus textbook is dropped from the top of a tall tower, the distance it has fallen after *t* seconds is given by the function $d(t)=16t^2$. Find the book's average speed over the following intervals:

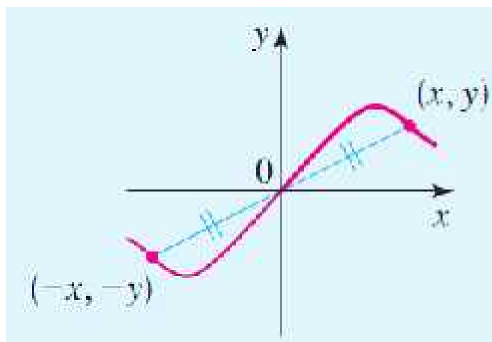
(a) $t \in [1,5]$ (b) $a \le t \le a+h$

If you've ever drawn a line straight down the middle of your body from the top of your head to the floor then looked in the mirror, you have realized how fun it can be to play with markers, but you've also realized that you posses a vertical line of bilateral symmetry. If you folded yourself along your drawn line, after a few "ouches," your right and left side would coincide nicely. Graphs have symmetries too!

- A function that has *y*-axis symmetry is called and **even function**. For such functions, $f(-x) = f(x)$ for all *x* in the domain of *f*.



- A function that has origin symmetry (180° rotational symmetry) is called an odd function. For such functions, $f(-x) = -f(x)$ for all *x* in the domain of *f*.



Example 6:
Determine algebraically if the following functions are even, odd, or neither. Verify graphically if possible.

(a) $f(x) = -x^2 + 5$ (b) $g(x) = x^2 - 4x + 3$ (c) $h(x) = 2x^3 - 3$

(d) $v(t) = -\frac{1}{2}t^3 - t$ (e) $r(m) = \frac{m^3}{3 - 2m^2}$ (f) $q(z) = \frac{3z^5 - 6z^3 + z}{3z^3 + 4z}$

(g) $y(x)=\dfrac{-2x^5+x^3+6}{3x^2+4}$

(h) $F(x)=-3\sqrt[3]{8x}$

(i) $t(d)=\dfrac{d^{2/3}}{\sqrt{d^2+4}}$

From the examples above, we can generalize some rules of thumbs about the symmetry of some types of functions based on the algebraic test of replacing x with $-x$.

For polynomials

- if NONE of the signs change, the function is ______________________________________.
- if ALL of the signs change, the function is ____.
- if SOME signs change and some stay the same, the function is _________________________.

For quotients of functions

- $\dfrac{\text{Even}}{\text{Even}}=$ ______________________________________.
- $\dfrac{\text{Even}}{\text{Odd}}$ or $\dfrac{\text{Odd}}{\text{Even}}=$ ______________________________.
- $\dfrac{\text{Odd}}{\text{Odd}}=$ ______________________________________.
- $\dfrac{\text{Neither}}{\text{whatever}}$ or $\dfrac{\text{whatever}}{\text{Neither}}=$ _________________________.

It helps to have these rules of thumb memorized, but it's never an excuse for not knowing how to perform the algebraic test. In fact, you will be asked to do just that on your exam!

Putting it all together so far:

Example 7:
For each of the following, find the domain, end behavior, *x*-intercepts, *y*-intercepts, any symmetry, and find and label all discontinuities. Use your collected information to sketch a graph.

(a) $f\left(x\right)=\dfrac{2x^2-2x-4}{3x^2+9x-30}$

(b) $g\left(x\right)=\dfrac{4x^2+8x}{x+2}$

(c) $h\left(x\right)=\begin{cases}3x-1, & x>2\\ 2x^2, & x\le 2\end{cases}$

(d) $p\left(x\right)=\begin{cases}x^2-1, & x<-1\\ 2, & x=0\\ -3x-3, & x>1\end{cases}$

(e) $h\left(x\right)=\begin{cases}\sqrt{x^2-4}, & \left|x\right|>2\\ 4-x^2, & \left|x\right|<2\end{cases}$