Chapter 2.5: Building Functions from other Functions

If you thought your fun with functions has reached a pinnacle, then you are sadly mistaken. Hopefully you will have "being mistaken" figured out by the next test.

Just as you can plug a toaster into an electrical outlet, a log into a wood chipper, an elephant into a spaceship, or even an entire universe into that same universe (whoooooooah!), you can also plug one function into itself. This is called **composition**, and you KNOW how to do this already.

Example 1:
You are buying a new elephant blanket (for the ride on the spaceship) for \$100. The elephant blanket proprietor says it's your lucky day (he's never sold one before), and he offers you BOTH a \$10 AND a 10% discount, but YOU get to decide the order in which the discounts are taken. Does it matter? Let p be the purchase price, $D(p)$ be the final price using the 10% discount, $T(p)$ be the final price using the \$10 off.

Write an equation for the final costs C_1 and C_2 of the final price if both discounts are used in different orders.

Just as we can combine numbers together in a variety of ways using the rules of algebra, we can, too, combine functions using a similar algebra. The above is just one example of how we can build new functions (total blanket cost) from other functions (separate discounts).

The Algebra of Functions

If f and g are functions, then

- $(f \pm g)(x) = f(x) \pm g(x)$
- $(fg)(x) = f(x)g(x)$
- $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)},\ g(x) \neq 0$
- $(f \circ g)(x) = f(g(x))$

Example 2:

Let $f(x)=\dfrac{1}{x-3}$ and $g(x)=\sqrt{x}$. Find the domains of *f* and *g*, then find

(a) $f+g$ (b) $f-g$ (c) fg (d) $\dfrac{f}{g}$ (e) $\dfrac{g}{f}$ (f) $f\circ g$ (g) $g\circ f$ (h) $f\circ f$ (i) $g\circ g$.

Example 3:

Find $f\circ g\circ h$ if $f(x)=\dfrac{x}{x+1}$, $g(x)=x^{20}$, and $h(x)=x+5$

Going forward is one thing. Going backwards is yet another.

Example 4:

Decompose the function $h(x)=\sqrt[3]{x-8}$ into two non-trivial functions *f* and *g*, such that $h=f\circ g$. How many unique decompositions can you find?

If you've ever done something you wish you could undo, then you are alone. I'm kidding, of course. We all have done something we'd like to have back. Although we can't do that in our own lives, in the world of functions, this is sometimes possible. Let me show you.

Example 5:
I'm thinking of an integer between 4.5 and 5.5, and I'm selfish. I'm not going to tell you what it is. If you tell me to take my number and add three to it, then quadruple it, then subtract 4, then divide by two, I would give you THAT number. Assuming I did, how could you find my original number about which I was so secretive, guarded, and selfish?

Was that last example Magic? Mathmagic? Cheesy?

The point is, there was a process for working backwards from the original output to arrive at the original input, and this input was unique. This is because the function I used to "scramble" my number was **one-to-one**, making my cipher "invertible."

> If $f(x)$ is a one-to-one function over a domain, then it is **invertible**, meaning, it has an inverse function, $g(x)$, such that $g(x) = f^{-1}(x)$.

When a function is inverted, all the *x* and *y* information interchange. Consequently, an invertible function containing the point (a, b) means that its inverse will contain the point (b, a). This has several implications, besides $f(a) = b \Leftrightarrow f^{-1}(b) = a$. If $y = f(x)$ is 1:1 over a domain,

- to graphically find the graph of f^{-1}, reflect the graph of *f* over the line $y = x$
- to algebraically find the equation of f^{-1}, interchange *x* and *y* in the equation of *f*, then resolve for *y*.

Example 6:
If $f(x) = 4x - 2$, determine if $f(x)$ is one-to-one. If it is, sketch the graph of *f*, then graphically and algebraically find f^{-1}. Find $f(2)$ and $f^{-1}(6)$.

Example 7:

Let $g(x)$ be the inverse of $f(x) = -2\sqrt[3]{x-1} - 5$. Find (a) g, (b) $g \circ f$, and (c) $f \circ g$.

Theorem for Inverses

Two functions *f* and *g* are inverses if and only if $(f \circ g)(x) = x = (g \circ f)(x)$

Example 8:

Let $h(x)$ be the inverse of $g(x) = \dfrac{2x-1}{4x+3}$. Find *h*, then verify, using the theorem above, to verify you have found the correct inverse. Give the domain of both *h* and *g*.