Chapter 3.1: Polynomial Functions

In Algebra I and Algebra II, you encountered some very famous polynomial functions. In this section, you will meet many other members of the polynomial family, what sets them apart from other families of functions, and will learn how to uncover deep secrets about many family members just by looking at their equation.

Formal Definition of a Polynomial Function

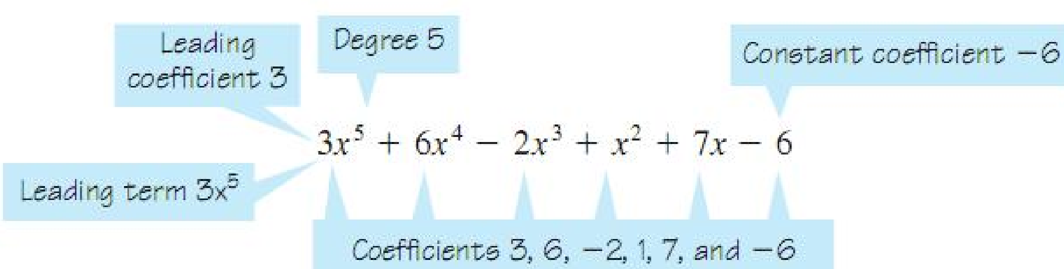
A **polynomial function** is a function whose domain is all real numbers that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \text{, such that } a_n \neq 0.$$

The above form, with all its sub- and super-scripts, is called the **standard form** of a polynomial equation. Let's dissect the equation and learn some taxonomy.

- n
  - represents the degree or order of the polynomial equation (the largest exponent in the equation)
  - $n \in W = \{0,1,2,3,\ldots\}$ (THIS IS WHY THE DOMAIN IS ALL REAL NUMBERS!)
- $a_i x^i$
  - represents a term of the polynomial
  - i is an index, used to keep track of the terms
- a_i
  - represents all the coefficients of the individual terms
  - $a_i \in \Re$
  - a_n represents the **leading coefficient,** $a_n \neq 0$
  - a_0 represents the **constant**
- x represents the independent variable of interest
- $f(x)$ represents the dependent variable of interest (interchangeable with y)
- f represents the name of the function

Here's what a polynomial looks like with numbers filled in—a much more comfortable look.



Example 1:
Identify each of the following as a polynomial (P) or not a polynomial (N). Be sure to understand why.

(a) $f(x)=4x^2+(\ln 3)x-1$ (b) $f(x)=10x^2-11x^3+3^x$ (c) $f(x)=3x^{2/3}+x^5-9^{-2}$

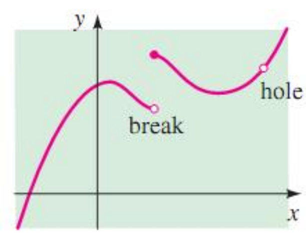
(d) $f(x)=x\left(2x^2-3x\right)$ (d) $f(x)=\dfrac{4}{x^2}-7x+x^2$ (e) $f(x)=\pi x^3+7x+ex^2+\ln 1$

(f) $f(x)=3\sqrt{x}-7\sqrt[3]{x}$ (g) $f(x)=\dfrac{x^3}{x}+\dfrac{2}{x^{-2}}+\left(\sqrt{x}\right)^2+\pi^0$

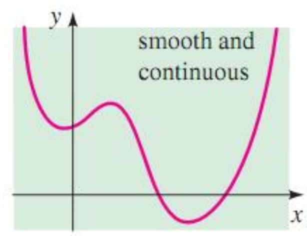
We can classify different types of polynomials by their degree. The graphs of polynomials of degree 0 are constants (horizontal lines), degree 1 graphs are linear (slanted), and degree 2 graphs are parabolas. In general, the greater the degree, the more complicated and sophisticated the graph will be. One thing is for sure, the graph of any polynomial function is always a smooth, continuous curve, meaning it has no breaks, jumps, gaps, chasms, VA's, etc or cusps and sharp turns or other dangerous, pointy points.

Example 2:
Identify each of the following as a graph of a polynomial (P) or not a polynomial (N).

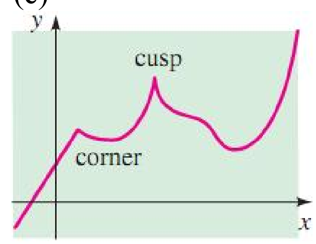
(a)



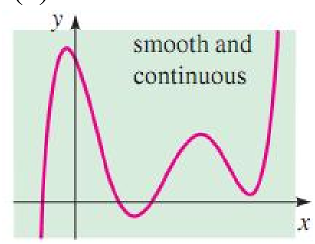
(b)



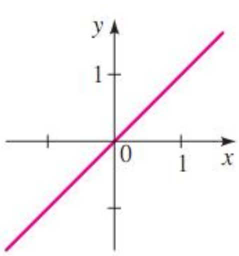
(c)



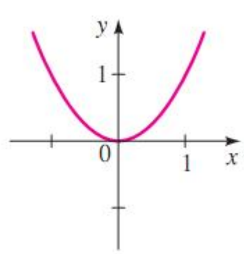
(d)



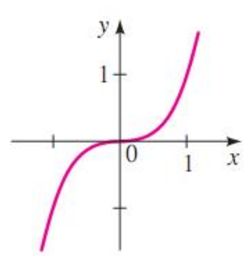
The simplest form of polynomial functions of various degrees are the single-termed polynomials, or **monomials**, of the form $f(x)=x^n$. You can think of these as the parent functions for all polynomials of degree n. For these monomials, there is a consistent pattern in the shapes of the graphs. In general, as the degree increases, the graphs become flatter between $|x|<1$ and steeper for $|x|>1$. Why is that??



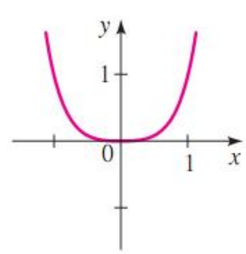
$y=x$
Linear



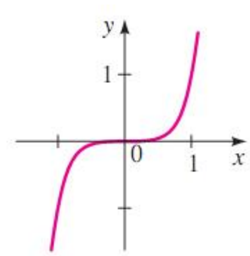
$y=x^2$
Quadratic



$y=x^3$
Cubic



$y=x^4$
Quartic



$y=x^5$
Quintic

Example 3:
Sketch the following transformations of the parent quartic function. Determine the end behavior of each graph, as well as the number of x-intercepts and relative extrema.

(a) $f(x)=(x-3)^4-2$

(b) $f(x)=-(x+2)^4-3$

(c) $f(x)=2x^4-8x^2$

(d) $f(x)=-3(x^2-1)(x^2-16)$

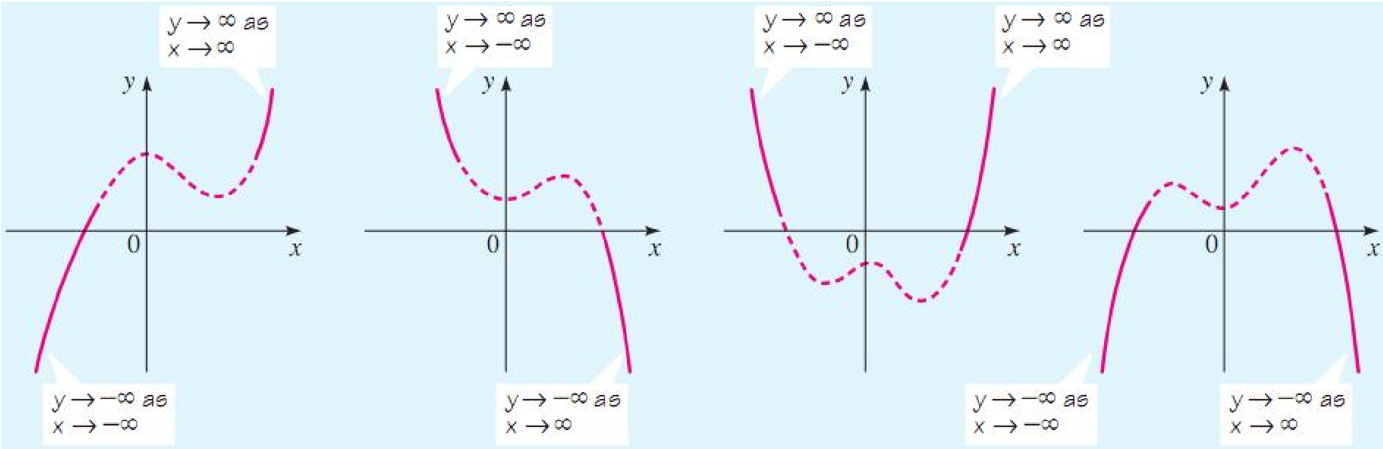
There is an easy way to determine the end behavior of a polynomial equation in standard form, $f(x)$, simply by looking at the leading term, $a_n x^n$. The sign of the leading coefficient will give you the right-end behavior. Do you know why this works?

- If $a_n>0$, then $\lim_{x\to\infty} f(x)=\infty$
- If $a_n<0$, then $\lim_{x\to\infty} f(x)=-\infty$

Once the right-end behavior is known, knowing whether the degree is even or odd will give you the left-end behavior.

- If n is even, then $\lim_{x\to-\infty} f(x)=\lim_{x\to\infty} f(x)$
- If n is odd, then $\lim_{x\to-\infty} f(x)=-\lim_{x\to\infty} f(x)$

Here's a visual:



n is odd	n is odd	n is even	n is even
$a_n > 0$	$a_n < 0$	$a_n > 0$	$a_n < 0$

Example 4:
Determine the end behaviors of the following polynomials:

(a) $f(x) = -2x^4$

(b) $P(x) = 5x^3 - 3x - 8 - 2x^4$

(c) $g(t) = 2t^7$

(d) $N(t) = 2t^7 + 4 - 6t^6 + 11t^3 - 8t$

In **Example 3**, we noticed that different equations of quartic polynomials had different number of relative extrema as well as a different number of x-intercepts. Information about both relative extrema and roots (zeros/x-intercepts) can be obtained from the degree of a polynomial equation.

Theorem

A polynomial function of degree n has at most n zeros and at most $n-1$ relative extrema.

This theorem gives you upper bounds on both possibilities. Any polynomial can have fewer than the maximum, and we have to figure out what those other possibilities are.

Example 5:
Draw several possibilities for the shape of a polynomial of odd degree. Use a degree 5 polynomial as one such function. Analyze the possibilities for zeros and relative extrema.

Example 6:
Draw several possibilities for the shape of a polynomial of even degree. Use a degree 4 polynomial as one such function. Analyze the possibilities for zeros and relative extrema.

In all the examples above, do you notice the relationship between the degree, the number of relative extrema, and the number of wiggles (inflection points)?

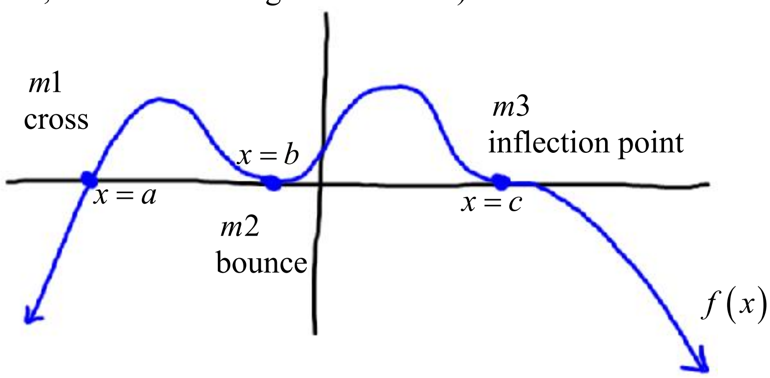
Before we summarize what we've learned, we need to talk about repeated roots of polynomial functions.

When a zero repeats itself d times, we say that zero has a **multiplicity of *d***, or **(*md*)**. The sum of the multiplicities of all the roots of a polynomial will equal the degree of the polynomial!

In this class, you are responsible for identifying graphically, and from factored forms of polynomials, single roots ($m1$), double roots ($m2$), and triple roots ($m3$).

Multiplicities in graphical and factored form

A polynomial $f(x)$ in factored form $f(x)=A(x-a)(x-b)^2(x-c)^3$, will have roots of $x=a$ ($m1$), $x=b$ ($m2$), and $x=c$ ($m3$). The degree of this polynomial will be $1+2+3=6$. A possible graph would look like this (of course, here, A would be a negative number.)



A direct consequence of this is a very, very, very important theorem called the **Factor Theorem**.

The Factor Theorem

$(x-a)$ is a factor of a polynomial function **if and only if** $x=a$ is a root of the same polynomial function.

Example 7:

Sketch a graph of the following function: $f(x)=-5x(x-3)^2(x+2)^3(x+5)^2(x-6)$. State the leading term and the degree of the polynomial.

Example 8:

Let $P(x) = -2x^4 - x^3 + 3x^2$. Find the zeros of *P*, then sketch the graph of *P*.

Example 9:

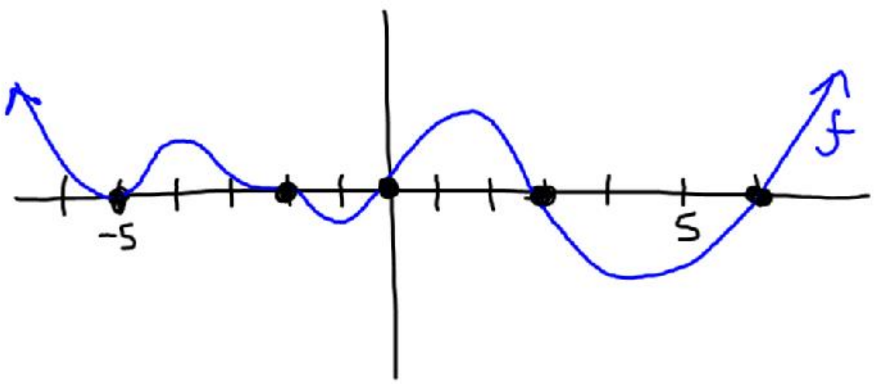
Let $h(m) = m^3 - 2m^2 - 4m + 8$. Find the roots of *h* using factor-by-grouping, then sketch the graph of *h*.

Example 10:

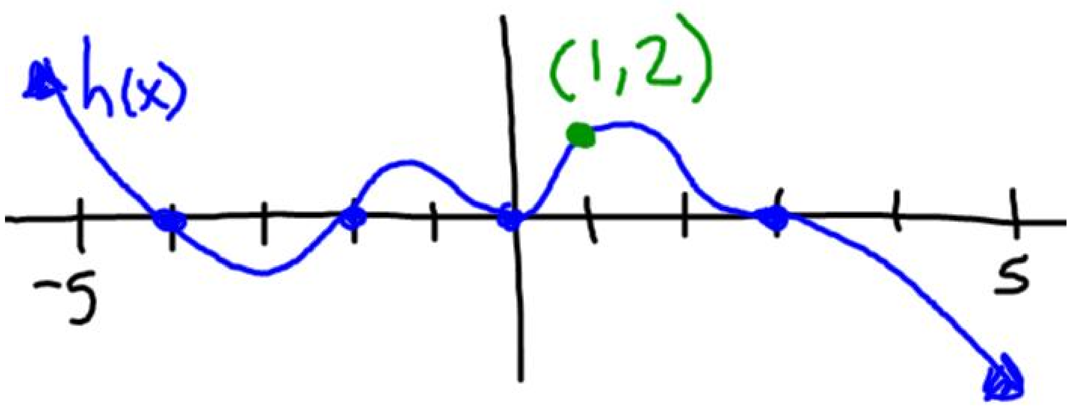
Write the (a) general equation of the polynomial function whose only roots are $x = 5\,(m1)$, $x = -1\,(m2)$, $x = 2\,(m3)$. Write the particular equations of the polynomial function from part (a) that passes through each the following points: (i) $(0,10)$ (ii) $(1,-4)$ HINT: It might help to sketch each first!

Example 11:
Write the particular equation of a degree 2 polynomial, f, with the following properties: $f(-3) = f(4) = 0$, $f(1) = 5$.

Example 12:
Write a general equation in factored form of the following graph:



Example 13:
Write a (a) general and (b) particular equation in factored form for the following graph:



Now I think we can summarize all we've learned about polynomial functions. First, let's start with what ALL polynomial functions have in common:

Important Chart I: What ALL Polynomials of degree *n* have in common

- Fun, oh so much fun
- Domain of all real numbers
- Smooth and continuous graphs
- Have a single *y*-intercept
- Have maximum *n* *x*-intercepts/roots/zeros
- Have up to $n-1$ relative extrema, occurring in pairs after that
- Have exponents that are Whole Numbers
- Infinite end behaviors
- No Asymptotes
- To determine the degree of a polynomial with all real roots, add the multiplicities of all the real roots.

Important Chart II: How Polynomials of degree *n* differ by whether *n* is even or odd

If *n* is EVEN	**If *n* is ODD**
• Same end behavior ○ $a_n > 0$, top in, top out ○ $a_n < 0$, bottom in, bottom out • Odd number of relative extrema, up to $n-1$ • No guarantee of a zero • Number of possible roots: 0, 1, 2, . . ., *n* • Bounded *y*-values (restricted range)	• Opposite end behavior ○ $a_n > 0$, bottom in, top out ○ $a_n < 0$, top in, bottom out • Even number of relative extrema up to $n-1$ • Guarantee of at least one real root • Number of possible roots: 1, 2, 3, . . ., *n* • Unbounded *y*-values (Range of all real numbers)