Chapter 3.2: Real Zeros of Polynomial Functions

In the last section, we found zeros of polynomial functions that were nice integers. There are other types of roots of polynomial equations that are still very real, but not very integery. Before we proceed with what types these are and how to find them, it's a good time to clarify some mathematical synonyms.

Mathematical synonyms

For a polynomial function $y = f(x)$, the following are equivalent;

1. Zeros of $f(x)$
2. Roots of $f(x)$
3. x-intercepts of the graph of $f(x)$
4. Solutions to the equation $f(x) = 0$

Let's look at an example that will illustrate other types of real roots.

Example 1:
Find the simplified, exact solutions to the following equation: $(2x-5)(x^2-5)=0$

The above example gives us a rational solution as well as two **conjugate** irrational solutions. We are now going to look into how we can go about finding these type of real roots, if and when they exist.

Rational Root Theorem

If the polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ as integer coefficients, then IF $f(x)$ happens to have rational roots, then they will be of the form $\frac{p}{q}$, where

- p is a factor of the constant, a_0, and
- q is a factor of the leading coefficient, a_n

This theorem does NOT guarantee rational solutions, only gives you a way to find possible candidates if they happen to exist. It could be that all the solutions are irrational or non-real.

Example 2:
Find the rational roots of $f(x)=x^3-3x+2$. Be sure to list all distinct, possible rational roots, then try them out individually.

Example 3:
List the distinct, possible rational roots of $g(x)=2x^3+x^2-13x+6$. Use your calculator to help you decide which, if any, of the candidates are actual zeros.

If we were to plug each of these candidates into the above equation by hand, it might take a while. There is an easier way to do this, and it happens to be the way most computer programs use to evaluate functions. It is called **nested form**.

Example 4:
Put the equation $g(x)=2x^3+x^2-13x+6$ into nested form, then evaluate $g(1)$, $g(2)$, $g(-3)$, and $g\left(\frac{1}{2}\right)$ using nested form.

As handy as nested form is, an astute person can simplify the substitution process even more my noticing the pattern used when evaluating a function that is in nested form. This streamlined process is known as **synthetic substitution.**

Example 5:

If $g(x) = 2x^3 + x^2 - 13x + 6$, evaluate $g(1)$, $g(2)$, $g(-3)$, and $g\left(\frac{1}{2}\right)$ using synthetic substitution.

If we look at the last box in the synthetic substitution process when substituting $x = a$ into a function $f(x)$, we know this is the desired function value, $f(a)$. When $f(a) = 0$, we know that $x = a$ is a root/zero/*x*-intercept of $f(x)$. By the **factor theorem**, we then know that $x - a$ is a factor of $f(x)$. This allows us to think of the process of synthetic substitution as a **synthetic division** process.

Division Algorithm (Specific)

When a polynomial function $f(x)$ is evaluated at $x = a$, it is equivalent to dividing $f(x)$ by the factor $x - a$. $f(a)$ is both the function value of *f* at $x = a$, but also the **remainder** when $f(x)$ is divided by $x - a$.

When $f(a) = 0$, $x - a$ is a factor of $f(x)$.

Example 6:

If $f(x) = x^4 - 13x^2 - 3x^3 + 15x$ and given that $f(-3) = 0$, find all real zeros, then write $f(x)$ as a product of linear factors.

Example 7:
Using both long and synthetic division, find all the zeros of $h(x) = 3x^3 - 12x^2 + 3x + 18$, given that $x + 1$ is a factor of $h(x)$.

Example 8:
If $f(x) = 9x^3 + 15x^2 - 77x - 147$, list all distinct, possible rational roots, then find the remaining roots given that $3x + 7$ is a factor of $f(x)$. Sketch the graph.

Example 9:
Find all real zeros of $f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$. List all distinct, possible rational roots. Use synthetic division test your possible roots.

Two of the roots in the previous example were irrational and were conjugate pairs of each other. If the real roots of a polynomial are real and not rational (or integer), then they must be irrational. There is an important theorem that relates to the pairing of such roots.

Radical conjugate Theorem

If $f(x)$ is a polynomial with **rational coefficients**, then if $x = a + \sqrt{b}$ is a root of $f(x)$, then $x = a - \sqrt{b}$ is too!

Example 10:
Using synthetic division, find all real roots of $g(x) = x^4 + 3x^3 - 9x^2 - 15x + 20$, given that $x = \sqrt{5}$ is a root.

Example 11:
A function is said to be in **reduced-factored form** if it is factored as a product of linear factors, $(x-a)(x-b)\cdots$, and/or irreducible quadratic factors (no radicals or imaginary units visible), $\left(x^2-a\right)\left(x^2-b\right)\cdots$. Write an equation, in reduced factored form, of a polynomial, *f*, of lowest degree with the following properties: $f(-1) = f\left(-\sqrt{2}\right) = f\left(2+\sqrt{3}\right) = 0$ and $f(0) = 5$.

Example 12:
Find the remainder when $6x^{1000} - 17x^{562} + 12x + 26$ is divided by $x + 1$

When division is done and the remainder is NOT zero, then the **divisor** is NOT a factor of the **dividend**. We can still do the division, though.

Division Algorithm (General)

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exists unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

or equivalently

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

Example 13:
Divide (a) 13 by 4, and (b) $3x^5 + 5x^4 - 4x^3 + 7x + 3$ by $x + 2$. Write your results in two different, yet equivalent ways.

Example 14:

Divide $8x^4 + 6x^2 + 1 - 3x$ by $2x^2 + 2 - x$. Write your results in two different, yet equivalent ways.

Example 15:

If $x + 2$ is a factor of $f(x) = 3x^3 + kx^2 + x - 2$, find the value of k.