Chapter 4.1: Exponentials & Logistics

Name something that grows fast!

Lay people tend to claim that ANYTHING that grows quickly exhibits exponential growth. This, of course, is not always the case. Be the person who knows what actually constitutes exponential growth.

Do you think the following scenario is exponential growth?
Scenario: You are offered $1 million today OR, you can wait 64 days and the total amount resulting from the following proposal. Start with a penny on the square of a checkerboard. Each day, the amount from the previous day is doubled and placed on the next square until all squares are filled. Which deal would you take?

Intuitive Definition of Exponential Growth or Decay

A quantity that increases or decreases exponentially will grow or decay, respectfully, at a rate proportional to the current amount.

Stated a different way: The more you have, ***the more you gain or, the less you have, the less you lose***. Things that grow or decay exponentially tend to increase or decrease by a **factor** or **percentage** each cycle.

Example 1:
List some real-life things that you know grow or decay exponentially. (There are no wrong answers here, only incorrect responses. We'll discuss each response—why they are and aren't.)

Formal Definition of Exponential Growth or Decay

An exponential function is of the form $f(x) = A \cdot b^x$, $A \neq 0$, where $b > 1$ or $0 < b < 1$.

b is called the **base**, where $b \in$ °
A is the vertical dilation factor, the **initial value**, and the y-intercept of the graph of $f(x)$.

What would happen if $b = 1$? If $b < 0$? If $A = 0$?

Here are some example of exponential functions:

$$f(x) = 3^x \qquad h(x) = 2 \cdot 10^x \qquad g(x) = 4\left(\frac{1}{2}\right)^x$$

Example 2:
Determine if each of the functions is exponential (E) or not exponential (N).

(a) $f(x) = 7^x$ (b) $f(x) = x^7$ (c) $f(x) = x^x$ (d) $f(x) = -2 \cdot 15^{x+1}$ (e) $f(x) = 4\pi^x$

(f) $f(x) = 3(1^x)$ (g) $f(x) = -3^x$ (h) $f(x) = (-3)^x$ (i) $f(x) = 0^x$ (j) $f(x) = \dfrac{3}{4^{-x}}$

Example 3:

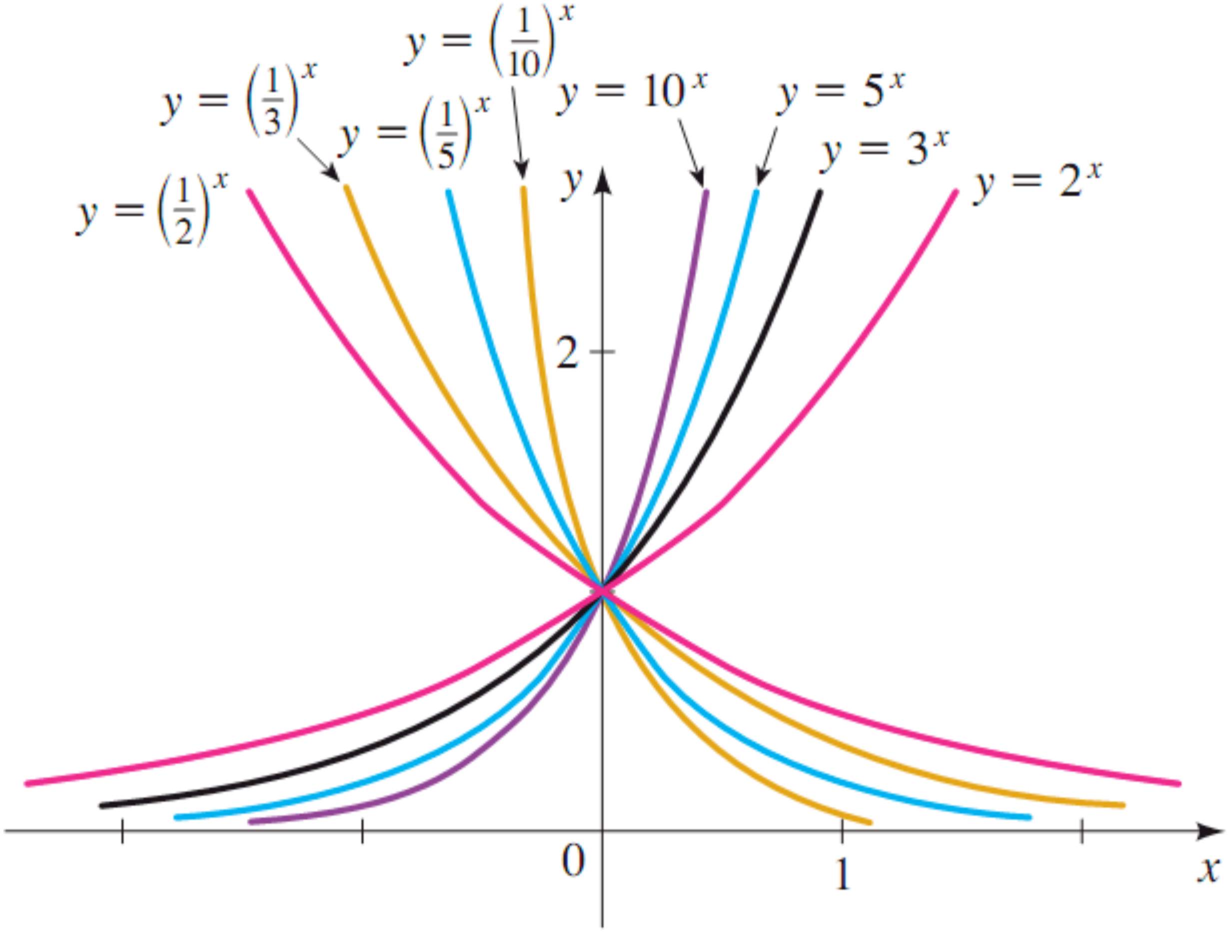
(Calculator) Let $f(x) = \left(\frac{1}{3}\right)^x$ and evaluate the following:

(a) $f(2)$ (b) $f\left(-\frac{3}{4}\right)$ (c) $f(\pi)$ (d) $f(\sqrt{2})$

Example 4:

Create a table of values and graph $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ on the same axes. What do you notice about them? Now do the same for $f(x) = e^x$ and $g(x) = \left(\frac{1}{e}\right)^x$. What do you notice?

The following graph shows the graphs of the family of exponential functions $f(x) = b^x$ for various values of the base b. Notice that for growth functions, the larger the base, the faster the growth for large x-values. For decay functions, the smaller the base, the faster the decay!



We can graph exponential functions by using our transformations.

Example 5:
Graph the parent function
$f(x) = e^x$. Find the domain and range, intercepts, symmetry, intervals of increasing/decreasing, and describe the end behavior.

Example 6:
Sketch $f(x) = 3 - e^{x-1}$. Give the domain and range. Find the exact value of the *y*-intercept. Describe the end behavior.

Example 7:
Sketch $f(x) = 4 - 3 \cdot 7^{5-x}$. Give the domain and range. Find the exact value of the *y*-intercept. Describe the end behavior.

Example 8:
A particular transformation of an exponential function can be achieved in more than one way. For each of the following, describe two different ways to achieve the graph of the given ending function as a transformation of the starting parent function.

(a) Start: $f(x) = x^2$, End: $g(x) = (4x)^2$

(b) Start: $f(x) = \sqrt{x}$, End: $g(x) = 4\sqrt{x}$

(c) S: $f(x) = 3^x$, E: $g(x) = 3^{x-1}$

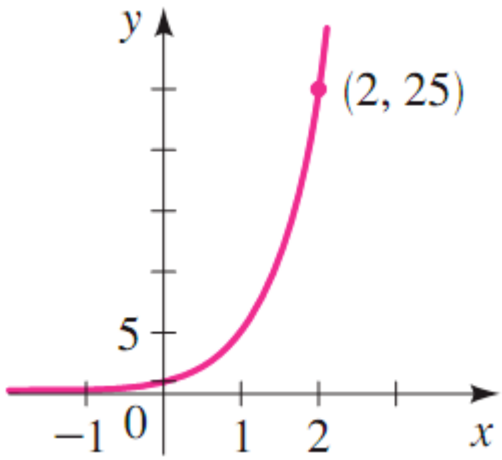
(d) S: $f(x) = 3^x$, E: $g(x) = 9^x$

(e) S: $f(x) = 3^x$, E: $f(x) = 9^{x+1}$

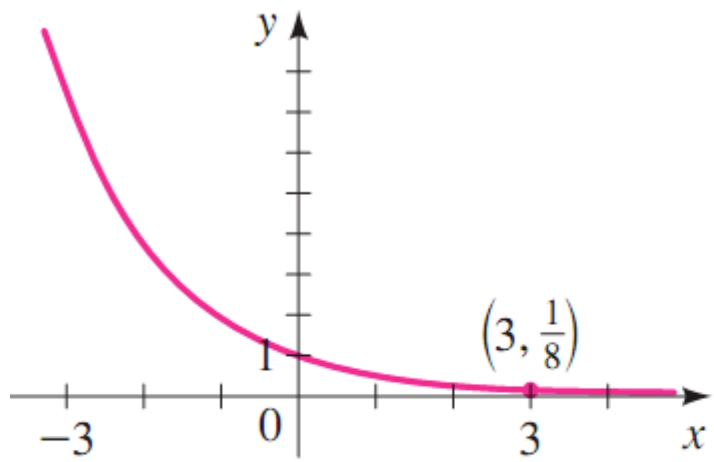
Example 9:

Find the exponential function $f(x) = b^x$ whose graph is given.

(a)



(b)



Example 10:

Write the equation of the exponential functions of the form $f(x) = A \cdot b^x$ that passes through the following points.

(a) $(0,4)$ and $(5,22)$

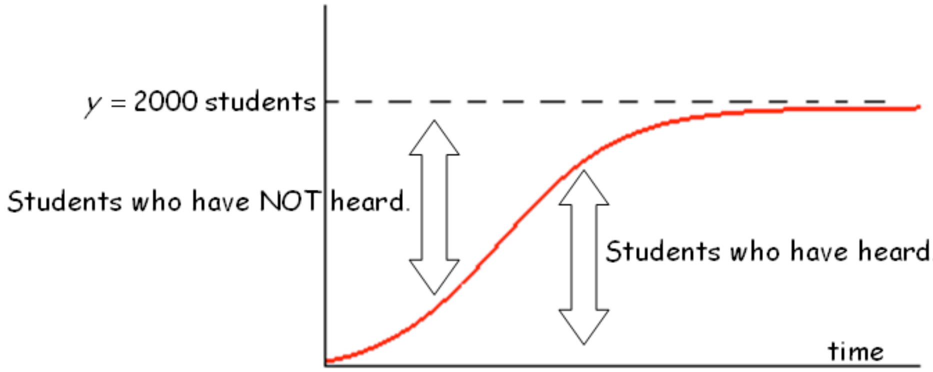
(b) $(0,2)$ and $\left(3,\frac{1}{2}\right)$

(c) $(-2,10)$ and $\left(2,\frac{1}{3}\right)$

(d) $\left(-5,\frac{1}{8}\right)$ and $(4,8)$

Not everything that grows exponentially continues to do so indefinitely. A special type of modified exponential growth function is called a **logistic function**.

The curve for the spread of a rumor might look like something shown below. The growth rate of the rumor is jointly proportional to the number of students who have heard and the number of students who have not heard the rumor.



Example 11:
(Calculator Permitted) A horrible rumor begins to spread on The NBHS campus (student population of 200). Carlos, Petunia, and Wallace begin telling people that "Titanium necklaces don't increase energy and balance!" After *t* days, the number of persons who have heard this dastardly rumor is modeled by the logistic function

$$R(t) = \frac{2000}{1+115e^{-0.97t}}$$

(a) How many people on campus originally heard the rumor? Round to the nearest whole person.
(b) To the nearest whole person, find the number of people who have heard the rumor after one day, two days, and five days.
(c) How many people have heard the rumor when the rumor is growing the fastest? On which day did this occur?
(d) Describe the end behavior of the graph. What does this mean in terms of the rumor?