Chapter 5.7: Inverse Trig Functions

Before we get full-fledged into inverse trig functions, it's worth reviewing trig functions, their inputs and outputs.

$$f(x) = \underbrace{\sin \overbrace{x}^{\text{angle}}}_{\text{ratio}}$$

***We take trig functions of angles to find ratios! We take trig functions of angles to find ratios!**

We have already had a little experience with an inverse trig **operation**.

Example 1:
If $\sin\theta = \frac{5}{7}$, solve for $\theta \in [0°, 360°)$

Notice that the equation above had TWO solutions within one positive rotation and INFINITELY MANY if you count coterminal angles and multiple rotations. This could be a problem for defining inverse trig functions.

$$f(x) = \underbrace{\sin^{-1} \overbrace{x}^{\text{ratio}}}_{\text{angle}}$$

If we try to evaluate the function above for $f\left(\frac{5}{7}\right)$, we get more than one solution, which violates the rules for a function! We are going to have to figure out a way to remedy that.

Recall the implications of a one-to-one function.

If f is a **one-to-one function** with domain A and range B, then its inverse f^{-1} is the function with the domain B and range A defined by

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

A function is one-to-one if it passes the horizontal line test. The graph of the inverse of such a function can then be obtained by reflecting the one-to-one function across the line $y = x$.

Unfortunately, none of the six trig functions are one-to-to. Luckily, we can still talk about their inverse functions if we can agree upon a convenient one-to-one interval in which to restrict them prior to generating their inverse graph.

Example 2:
Draw a couple of cycles of $f(x) = \sin x$, then find a convenient one-to-one interval that contains all the sine ratios in the interval $-1 \le \sin x \le 1$. Reflect this portion across the line $y = x$ to generate the graph of the inverse function. Give the domain and **principal value range** of this new inverse function.

The inverse sine function is the function $f(x) = \sin^{-1} x$ with domain $[-1,1]$ and **principal value range** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This means that x represents a sine ratio and $f(x)$ represents an acute angle in either quadrants I or IV.
The inverse sine function is also called the **arcsine function, denoted by $f(x) = \arcsin x$

It is helpful to think of $\sin^{-1} x$ or $\arcsin x$ as **"the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x."**

Example 3:
If $f(x) = \sin^{-1} x$, evaluate the following using the Unit Circle. Give your answer in radians.

(a) $f\left(\frac{1}{2}\right)$ (b) $f\left(-\frac{1}{2}\right)$ (c) $f\left(-\frac{\sqrt{3}}{2}\right)$ (d) $f(1)$ (e) $f(\pi)$

Example 4:
If $f(x) = \arcsin x$, using a calculator, evaluate the following to 3-decimal places in radians.

(a) $f(0.82)$ (b) $f\left(-\frac{1}{3}\right)$ (c) $f\left(\frac{3}{2}\right)$

*Sometimes, to denote the difference between the inverse sine operation, versus the inverse sine function, a **capital letter** is used to denote the inverse sine function.

Example 5:
Find the exact value of the following:

(a) $\cos\left(\text{Sin}^{-1}\frac{3}{5}\right) =$ (b) $\tan\left(\text{Arcsin}\frac{3}{5}\right) =$ (c) $\csc\left(\text{Sin}^{-1}\frac{3}{5}\right) =$

*In general, when finding angles in unspecified intervals, we assume we are finding angles of the inverse trig function in the principal value range.

From the properties of inverse functions, we get the following important relation.

$$\sin(\arcsin x) = x \quad \text{for } -1 \le x \le 1$$

and

$$\arcsin(\sin x) = x \quad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Example 6:
Evaluate the following compositions without a calculator:

(a) $\sin\left(\sin^{-1}\left(-0.0316\right)\right)=$ (b) $\sin^{-1}\left(\sin\frac{\pi}{4}\right)=$ (c) $\arcsin\left(\sin\frac{3\pi}{4}\right)=$

(d) $\arcsin\left(\sin\frac{13\pi}{10}\right)=$ (e) $\sin\left(\text{Sin}^{-1}\left(\frac{4}{3}\right)\right)=$

Example 7:
Draw a couple of cycles of $f\left(x\right)=\cos x$, then find a convenient one-to-one interval that contains all the cosine ratios in the interval $-1\le\sin x\le 1$. Reflect this portion across the line $y=x$ to generate the graph of the inverse function. Give the domain and **principal value range** of this new inverse function.

The inverse cosine function is the function $f\left(x\right)=\cos^{-1}x$ with domain $\left[-1,1\right]$ and **principal value range** $\left[0,\pi\right]$. This means that x represents a cosine ratio and $f\left(x\right)$ represents a positive angle, either acute or obtuse in either quadrants I or II.
The inverse cosine function is also called the **arccosine function, denoted by $f\left(x\right)=\arccos x$

Example 8:

Find (a) $\cos^{-1}\frac{\sqrt{3}}{2}$ (b) $\text{Arccos}\left(-\frac{1}{2}\right)$ (c) $\cos^{-1}0$ (d) $\arccos\frac{5}{7}$

From the properties of inverse functions, we get the following important relation.

$$\cos(\arccos x) = x \quad \text{for } -1 \le x \le 1$$
and
$$\arccos(\cos x) = x \quad \text{for } 0 \le x \le \pi$$

Example 9:
Evaluate the following compositions without a calculator:

(a) $\text{Cos}^{-1}\left(\cos\frac{3\pi}{4}\right) =$

(b) $\arccos\left(\cos\frac{7\pi}{6}\right) =$

(c) $\cos^{-1}\left(\cos\frac{\sqrt{2}}{2}\right) =$

(d) $\cos(\text{Arccos}(-0.5)) =$

(e) $\cos^{-1}\left(\cos\frac{23\pi}{7}\right) =$

Example 10:
Draw a couple of cycles of $f(x) = \tan x$, then find a convenient one-to-one interval that contains all the tangent ratios in the interval $-\infty \le \tan x \le \infty$. Reflect this portion across the line $y = x$ to generate the graph of the inverse function. Give the domain and **principal value range** of this new inverse function.

The inverse tangent function is the function $f(x) = \tan^{-1} x$ with domain $(-\infty, \infty)$ and **principal value range** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This means that *x* represents a tangent ratio and $f(x)$ represents an acute angle, either positive or negative, in either quadrants I or IV.
The inverse tangent function is also called the **arctangent function, denoted by $f(x) = \arctan x$

Example 11:

Find (a) $\tan^{-1} 1$ (b) $\arctan\left(-\frac{\sqrt{3}}{3}\right)$ (c) $\tan^{-1} 0$ (d) $\arctan 30$

From the properties of inverse functions, we get the following important relation.

$$\tan(\arctan x) = x \quad \text{for } -\infty < x < \infty$$
and
$$\arctan(\tan x) = x \quad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Example 12:
Evaluate the following compositions without a calculator:
(a) $\text{Tan}^{-1}\left(\tan\frac{7\pi}{4}\right) =$ (b) $\tan(\arctan(-1000.22)) =$ (c) $\arctan\left(\tan\left(-\frac{4\pi}{3}\right)\right) =$ (d) $\text{Tan}^{-1}\left(\tan\frac{19\pi}{15}\right) =$

Example 13:
Evaluate the following compositions without a calculator:
(a) $\sin\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right) =$ (b) $\text{Arccos}\left(\sin\frac{5\pi}{3}\right) =$ (c) $\tan^{-1}(\cos\pi) =$

(d) $\cot\left(\arcsin\left(-\frac{\sqrt{3}}{2}\right)\right) =$ (e) $\tan(\sin 2\pi) =$ (f) $\sec(\tan\pi) =$

(g) $\cos^{-1}(\arcsin 0) =$ (h) $\sin(2\arccos 0.5) =$ (i) $\sec\left(\arctan\left(\sin\frac{3\pi}{2}\right)\right) =$

We can also compose trig functions with inverse trig functions. The result is an algebraic expression that is equivalent for all values in the domain. This gives us an equation that is known as an **identity**.

Example 14:
Write the following as algebraic expressions.

(a) $\cos(\arctan x) =$ (b) $\sin(\cos^{-1} 2x) =$ (c) $\cot^2(\arccos x^2) =$

Of course, once we compose, we can venture on to decomposing!

Example 15:
Decompose the following algebraic expressions into the composition of a trig function with an inverse trig function. Answers may vary.

(a) $\frac{\sqrt{1+x^2}}{x} =$ (b) $\frac{3x}{\sqrt{1-9x^2}} =$ (c) $\frac{1}{3}\sqrt{9-16x^2} =$