Chapter 6.1: Fundamental Identities

There are many different types of mathematical equations, several of which we've already encountered this year.

Name equations—an equation that gives a name to a specific expression, for example $f(x) = x^{\sin x} - e^x$.

Conditional equations—equations that are meant to be solved and are only true for a finite number of values, for example $x^2 = 6x - 8$.

Contradictions—equations that equate two expressions that are never equal, for example $5 = 7$ or $\sqrt{1-x} = \ln x$.

Identities—equations that equate two expressions that are true for all values in the domain of both expressions, for example $7 + 1 = 10 - 2e^0$ or $|x| = \sqrt{x^2}$

Identities are important because the two equated expressions serve as mathematical synonyms of each other, meaning, one may be substituted for the other without loss of generality. We might want to do this, of course, to make the mathematical expressions we're working with easier to work with, but still equivalent.

We will begin by looking at some fundamental trigonometric identities, so called because they involve trigonometric expressions and are so basic that other, more sophisticated identities can be built from them and they can be similarly used to prove bigger, badder, bolder identities.

Reciprocal Identities

$\sin\theta = \dfrac{1}{\csc\theta}$ and $\csc\theta = \dfrac{1}{\sin\theta}$ $\quad$ $\cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$ $\quad$ $\tan\theta = \dfrac{1}{\cot\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$

Example 1:

Using the definitions of the trig functions in terms of x, y, and r, prove that $\tan\theta = \dfrac{1}{\cot\theta}$.

$\tan\theta \neq \dfrac{1}{\cot\theta}$

$\dfrac{1}{\frac{x}{y}}$

$1 \cdot \dfrac{y}{x}$

$\dfrac{y}{x}$

$\tan\theta$

Once an identity has been proven or established, it can be algebraically modified to take on a different form that would still be an identity. For instance, since $\sin\theta = \dfrac{1}{\csc\theta}$, we can say that $\sin\theta\csc\theta = 1$.

Quotient/Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

~~*The Reciprocal and Ratio Identities work for powers as well, for example~~

$\dfrac{1}{\csc^2\theta} = \sin^2\theta$ and $\tan^5\theta = \dfrac{\sin^5\theta}{\cos^5\theta}$

Example 2:

Using the definitions of the trig functions in terms of x, y, and r, prove that $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

$$\cot\theta \stackrel{?}{=} \frac{\cos\theta}{\sin\theta}$$

$$\frac{x/r}{y/r}$$

$$\frac{x}{r}\cdot\frac{r}{y}$$

$$\frac{x}{y}$$

$$\cot\theta$$

Pythagorean Identities (PIDs)

$$\cos^2 x + \sin^2 x = 1 \qquad 1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

*We can use x in place of θ, and typically do so if the angles are in radians.

** $\cos^2 x$ is a condensed way to write $(\cos x)^2$. The entire ratio is squared.

Example 3:

Using the definitions of the trig functions in terms of x, y, and r, prove that $\cos^2 x + \sin^2 x = 1$.

$$\cos^2 x + \sin^2 x \stackrel{?}{=} 1$$

$$(\cos x)^2 + (\sin x)^2$$

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$$

$$\frac{x^2}{r^2} + \frac{y^2}{r^2}$$

$$\frac{x^2+y^2}{r^2}$$

$$\frac{r^2}{r^2}$$

$$1$$

Even/Odd Identities

$\sin(-x) = -\sin x$	$\tan(-x) = -\tan x$	$\cos(-x) = \cos x$
$\csc(-x) = -\csc x$	$\cot(-x) = -\cot x$	$\sec(-x) = \sec x$
ODD	ODD	EVEN

Cofunction Identities

$\sin x = \cos\left(\frac{\pi}{2} - x\right)$	$\tan x = \cot\left(\frac{\pi}{2} - x\right)$	$\sec x = \csc\left(\frac{\pi}{2} - x\right)$
$\cos x = \sin\left(\frac{\pi}{2} - x\right)$	$\cot x = \tan\left(\frac{\pi}{2} - x\right)$	$\csc x = \sec\left(\frac{\pi}{2} - x\right)$

Before we start using these new fantastic identities to prove other identities and solve conditional equations, we need to learn the mathematical difference between **verification** and **proof** which, outside a mathematical context, can be seen as equivalent. Mathematically they are very different.

Verification—using concrete, actual, finite values in a case-by-case scenario to show that two expressions are equivalent for the specific values being used.

Proof—showing that two expressions are equivalent in ALL cases. The key is to use variables rather than specific values.

<u>NO AMOUNT OF VERIFICATION CONSTITUTES A MATHEMATICAL PROOF!</u>

Example 4:

For both $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{4}$, verify that $\tan x = \cot\left(\frac{\pi}{2} - x\right)$

$\tan\frac{\pi}{6} \stackrel{?}{=} \cot\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$

$\frac{\sqrt{3}}{3}$ | $\cot\left(\frac{2\pi}{6}\right)$

$\cot\left(\frac{\pi}{3}\right)$

$\frac{\sqrt{3}}{3}$ ✓

$\tan\frac{7\pi}{4} \stackrel{?}{=} \cot\left(\frac{\pi}{2} - \frac{7\pi}{4}\right)$

-1 | $\cot\left(-\frac{5\pi}{4}\right)$

$\cot\left(\frac{3\pi}{4}\right)$

-1 ✓

A **counterexample** is a specific example that shows a claim to be false. While no amount of verification constitutes a mathematical proof, it only takes one counterexample to disprove a claim.

Example 5:
Find a counterexample to disprove each of the following "identities."

(a) $\sqrt{x^2+y^2} \neq x+y$

$\boxed{x=1, y=1}$ $\sqrt{1^2+1^2}$ | $1+1$

$\sqrt{2} \neq 2$

(b) $\sin(x+y) \neq \sin x + \sin y$

$\boxed{x=\frac{\pi}{4},\ y=\frac{\pi}{4}}$ $\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)$ | $\sin\frac{\pi}{4}+\sin\frac{\pi}{4}$

$\sin\frac{\pi}{2}$ | $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$

$1 \neq \sqrt{2}$

(c) $\cos x + \sin x \neq 1$ $x=\frac{\pi}{4}$

$\cos\frac{\pi}{4}+\sin\frac{\pi}{4}$ | 1

$\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$ | 1

$\sqrt{2} \neq 1$

Example 6:
Simplify the following trig expressions to a power of a single trig function:

(a) $\sin^3 x + \sin x\cos^2 x$

$\sin x(\sin^2 x + \cos^2 x)$

$\sin x\ (1)$

$\sin x$

– or –

$\sin^3 x + \sin x(1-\sin^2 x)$

$\cancel{\sin^3 x} + \sin x - \cancel{\sin^3 x}$

$\sin x$

(b) $(\sec x - 1)(\sec x + 1)$

$\sec^2 x + \sec x - \sec x - 1$

$(\sec^2 x) - 1$

$(1+\tan^2 x) - 1$

$\tan^2 x$

(c) $\left(\frac{1}{\cos\theta}-1\right)\left(\cos^2\theta + \sec(-\theta) + \frac{1}{\csc^2\theta}\right)$

$(\sec\theta - 1)(\cos^2\theta + \sec\theta + \sin^2\theta)$

$(\sec\theta - 1)(1+\sec\theta)$

$(\sec\theta - 1)(\sec\theta + 1)$

$\sec^2\theta - 1$

$\tan^2\theta$

(d) $\frac{1-\sin^2 x}{\csc^2 x - 1}$

$\frac{\cos^2 x}{\cot^2 x}$

$\cos^2 x \cdot \tan^2 x$

$\cancel{\cos^2 x} \cdot \frac{\sin^2 x}{\cancel{\cos^2 x}}$

$\sin^2 x$

(e) $\frac{\cos x}{1-\sin x} - \frac{\sin x}{\cos x}$

$\frac{\cos^2 x - \sin x(1-\sin x)}{\cos x(1-\sin x)}$

$\frac{\cos^2 x - \sin x + \sin^2 x}{\cos x(1-\sin x)}$

$\frac{\cancel{(1-\sin x)}}{\cos x\cancel{(1-\sin x)}}$

$\frac{1}{\cos x}$

$\sec x$

(f) $\frac{-\tan\left(\frac{\pi}{2}-\beta\right)\csc(-\beta)}{\cot^2\beta + 1}$

$\frac{(-\cot\beta)(-\csc\beta)}{\csc^2\beta}$

$\frac{(\cot\beta)(\cancel{\csc\beta})}{(\csc\beta)(\cancel{\csc\beta})}$

$\frac{\cos\beta}{\cancel{\sin\beta}} \cdot \frac{\cancel{\sin\beta}}{1}$

$\cos\beta$

Simplifying expressions and/or rewriting them in more useful, equivalent expressions is very helpful when trying to algebraically solve trigonometric equations.

Example 7:

Find the solutions to $2\sin^2 x + \sin x = 1$, such that (a) $x \in [0, 2\pi)$, (b) $x \in [0, 4\pi)$, and (c) $x \in (-\infty, \infty)$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$2\sin x - 1 = 0$ or $\sin x + 1 = 0$

$\sin x = \frac{1}{2}$ or $\sin x = -1$

(a) $\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad x = \frac{3\pi}{2}}$

(b) $\boxed{x = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{3\pi}{2}, \frac{7\pi}{2}}$ (add 2π to each answer from (a))

(c) $\boxed{\begin{array}{l} x = \frac{\pi}{6} + 2\pi n, n \in \mathbb{Z} \\ x = \frac{5\pi}{6} + 2\pi n, n \in \mathbb{Z} \\ x = \frac{3\pi}{2} + 2\pi n, n \in \mathbb{Z} \end{array}}$

Example 8:

Find the solutions to $\frac{\cos^3 x}{\sin x} = \cot x$, such that $x \in [-2\pi, 2\pi]$. (Be careful not to lose any variable information in the process, like the $\cot x$!) Make sure your final answers are in the domain of the equation.

$\cos^2 x \cdot \frac{\cos x}{\sin x} - \cot x = 0$

$\cos^2 x \cdot \cot x - \cot x = 0$

$\cot x(\cos^2 x - 1) = 0$

$\cot x(-\sin^2 x) = 0$

$-\cot x \cdot \sin^2 x = 0$

So $\cot x = 0$ or $\sin^2 x = 0$

$\boxed{\begin{array}{l} x = \frac{\pi}{2}, -\frac{\pi}{2} \\ x = \frac{3\pi}{2}, -\frac{3\pi}{2} \end{array}}$

$\sin x = 0$

$\cancel{x = 0, -2\pi, 2\pi}$

$\cancel{x = \pi, -\pi}$

but $\sin x \neq 0$ or we divide by zero in the original equation, so we cast these out

Example 9:

Algebraically find ALL solutions: $\sqrt{2}\tan x\cos x = \tan x$

$\sqrt{2}\tan x\cos x - \tan x = 0$

$\tan x(\sqrt{2}\cos x - 1) = 0$

$\tan x = 0$ or $\cos x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$x = 0 + 2\pi n, n \in \mathbb{Z}$

$x = \pi + 2\pi n, n \in \mathbb{Z}$

or just

$\boxed{x = 0 + \pi n, n \in \mathbb{Z}}$

$\boxed{\begin{array}{l} x = \frac{\pi}{4} + 2\pi n, n \in \mathbb{Z} \\ x = \frac{7\pi}{4} + 2\pi n, n \in \mathbb{Z} \end{array}}$

Example 10:

Find all solutions such that $\psi \in [-4\pi, 0)$, then breathe a Psi of relief: $3\sin\psi = 2\cos^2\psi$

$3\sin\psi - 2(\cos^2\psi) = 0$

$3\sin\psi - 2(1 - \sin^2\psi) = 0$

$2\sin^2\psi + 3\sin\psi - 2 = 0$

$(2\sin\psi - 1)(\sin\psi + 2) = 0$

$\sin\psi = \frac{1}{2}$ or $\sin\psi = -2$ No solution

So $\sin\psi = \frac{1}{2}$

$\boxed{\psi = -\frac{11\pi}{6}, -\frac{23\pi}{6}, -\frac{7\pi}{6}, -\frac{19\pi}{6}}$

Example 11:

Rapid fire, mixed practice: Solve each of the following such that $0 \le x < 2\pi$. No calculator.

(a) $2\sin x + \sqrt{3} = 0$

$\sin x = -\frac{\sqrt{3}}{2}$

$\boxed{x = \frac{4\pi}{3}, \frac{5\pi}{3}}$

(b) $4\sec^2 x = 8$

$(\sec x)^2 = 2$

$\sec x = \pm\sqrt{2}$

$\cos x = \pm\frac{\sqrt{2}}{2}$

$\boxed{x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

(c) $4\cos^2 x - 3 = 0$

$(\cos x)^2 = \frac{3}{4}$

$\cos x = \pm\frac{\sqrt{3}}{2}$

$\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}}$

(d) $\sin x = \cos x$

(by thinking) $\boxed{x = \frac{\pi}{4}, \frac{5\pi}{4}}$

(or by algebra)

$\sin^2 x = \cos^2 x$

$\sin^2 x = 1 - \sin^2 x$

$2\sin^2 x = 1$

$\sin x = \pm\frac{\sqrt{2}}{2}$

$x = \frac{\pi}{4}, \cancel{\frac{3\pi}{4}}, \frac{5\pi}{4}, \cancel{\frac{7\pi}{4}}$

extraneous

(e) $2\cos^2 x = \cos x$

$2\cos^2 x - \cos x = 0$

$\cos x(2\cos x - 1) = 0$

$\cos x = 0, \cos x = \frac{1}{2}$

$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}, \frac{5\pi}{3}}$

(f) $2 + 2\cos x = \sin^2 x$

$2 + 2\cos x = 1 - \cos^2 x$

$\cos^2 x + 2\cos x + 1 = 0$

$(\cos x + 1)^2 = 0$

$\cos x = -1$

$\boxed{x = \pi}$