Chapter 6.2: Trig Proofs

Proofs are fun, simply because they can be so challenging. No two are alike. While there are several common strategies for analytically proofing non-fundamental trig identities, the sequence and the number of steps from one to the next is as varied as the music on someone's .mp3 player.

To succeed at proofs, one needs to have a sense of adventure and a fearless attitude. It also helps to know your trig identities by heart and to be an excellent practitioner of algebraically manipulating expressions. A

As we set off on this great adventure together, consider this quote by mathematician W.S. Anglin:

> *Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigor should be a signal to the historian that the maps have been made, and the real explorers have gone elsewhere.*

The methods used in proving an identity are very different from solving a conditional equation. Here are a few guidelines to follow:

1. It is helpful to draw a vertical line down through the equal sign of the identity to prove. This will encourage you to work with each side separately.
2. Begin by starting on the more "complicated" looking side, that is, the side with more information. Afterall, it is easier to "tear down" than to "build up."
3. Continue on one side making of the identity by making trig identity substitutions and/or any **algebraic techniques** until that side looks exactly like the other side.
4. If you get stuck on one side, jump to the other side and work on getting that expression equal to the other side, **<u>BUT ALWAYS KEEP YOUR EYE ON THE OTHER SIDE, AS IT BECOMES YOUR TARGET!!!!!!</u>**
5. The proof is over when the expression on the last line of one side looks exactly like the expression on the bottom line of the other side.
6. Sign you name on your masterpiece, or write "Q.E.D." at the bottom

Here's a toe-dipping example to get us started.

Example 1:
Prove the following identity:

$$\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$$

$$(\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)$$

$$(\sin^2 x - \cos^2 x)(1)$$

$$(\sin^2 x) - \cos^2 x$$

$$(1 - \cos^2 x) - \cos^2 x$$

$$1 - 2\cos^2 x$$

– Sizzle!

We will point out the algebraic morals of each example as we encounter them. From the previous example, we get the following, important algebraic strategy . . .

- **FACTORING (COMMON FACTORS, DIFFERENCE OF SQUARES, PERFECT SQUARE TRINOMIALS, DIFFERENCE/SUM OF TWO CUBES . . .)**

Example 2:
Prove the following identity:

$$\tan x + \cot x \stackrel{?}{=} \sec x \csc x$$

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$\frac{\sin^2 x + \cos^2 x}{\cos x \cdot \sin x}$$

$$\frac{1}{\cos x \cdot \sin x}$$

$$\frac{1}{\cos x} \cdot \frac{1}{\sin x}$$

$$\sec x \csc x$$

– Bam!

That last example had two juicy morsels:

- **WHEN ALL ELSE FAILS, GET EVERYTHING IN TERMS OF SINE AND COSINE**
- **COMBINE TERMS BY GETTING A COMMON DENOMINATOR**

Example 3:
Prove the following identity:

$$\frac{1}{\sec x - 1} + \frac{1}{\sec x + 1} \stackrel{?}{=} 2\cot x \csc x$$

$$\frac{(\sec x + 1) + (\sec x - 1)}{(\sec x - 1)(\sec x + 1)}$$

$$\frac{2\sec x}{\sec^2 x - 1}$$

$$\frac{2\sec x}{\tan^2 x}$$

$$\frac{2}{\tan x} \cdot \frac{\sec x}{\tan x}$$

$$2\cot x \cdot \frac{1}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\sin x}$$

$$2\cot x \cdot \frac{1}{\sin x}$$

$$2\cot x \csc x$$

– Yeppers!

Adding to our awesome list of algebraic strategies:

- **ELIMINATE COMPLEX FRACTIONS**

Example 4:
Prove the following identity:

$$\frac{\cos t}{1-\sin t} = \frac{1+\sin t}{\cos t}$$

$$\frac{1+\sin t}{\cos t}\left(\frac{1-\sin t}{1-\sin t}\right)$$

$$\frac{1-\sin^2 t}{\cos t(1-\sin t)}$$

$$\frac{\cos^2 t}{\cos t(1-\sin t)}$$

$$\frac{\cancel{\cos t}\cdot\cos t}{\cancel{\cos t}(1-\sin t)}$$

$$\frac{\cos t}{1-\sin t}$$

– Jingle, Jangle, Jingle

Another strategy:

- **MULTIPLY BY A CLEVER FORM OF ONE $\left(\frac{\text{CONJUGATE}}{\text{CONJUGATE}}\right)$, TO SET UP A DIFFERENCE OF SQUARES AND A PYTHAGOREAN SUBSTITUTION.**
- **EXPAND ONLY THE CONJUAGTES, LEAVING THE OTHER SIDE OF THE FRACTION FACTORED, READY TO DIVIDE OUT COMMON FACTORS.**

Example 5:
Prove the following identity:

$$\left(\frac{1-\cos\theta}{1-\cos\theta}\right)\frac{\sec\theta}{1+\cos\theta} = \csc^2\theta(\sec\theta-1)$$

$$\frac{\sec\theta-\cos\theta\sec\theta}{1-\cos^2\theta}$$

$$\frac{\sec\theta-1}{\sin^2\theta}$$

$$\csc^2\theta(\sec\theta-1)$$

– More, please!

Yet another strategy:

- **DO ANY "OBVIOUS" MATH, SUCH AS DISTRIBUTING, COMBINING LIKE TERMS, ETC.**

Example 6:
Prove the following identity:

$$\frac{\cot^2\Phi}{1+\csc\Phi} \stackrel{?}{=} \cot\Phi\left(\sec\Phi - \tan\Phi\right)$$

Left side:

$$\left(\frac{\sin^2\phi}{\sin^2\phi}\right)\frac{\frac{\cos^2\phi}{\sin^2\phi}}{1+\frac{1}{\sin\phi}}$$

$$\frac{\cos^2\phi}{\sin^2\phi+\sin\theta}$$

Right side:

$$\cot\phi\sec\phi - \cot\phi\tan\phi$$

$$\frac{\cos\phi}{\sin\phi}\cdot\frac{1}{\cos\phi} - 1$$

$$\frac{1}{\sin\phi} - 1$$

$$\frac{1-\sin\theta}{\sin\theta}\left(\frac{1+\sin\theta}{1+\sin\theta}\right)$$

$$\frac{1-\sin^2\theta}{\sin^2\theta+\sin\theta}$$

$$\frac{\cos^2\theta}{\sin^2\theta+\sin\theta}$$

–Phew!

Kernels to extract:

- **CHANGE THE VARIABLES IF YOU DON'T LIKE THE GIVEN ONES**
- **WORK ON BOTH SIDES AND "MEET IN THE MIDDLE"**

That concludes the major list of algebraic methods. You may develop others as you work more and more and more and more problems. Here's your chance.

Example 7:
Prove the following identity:

$$\frac{1+\cos x}{1-\cos x} \stackrel{?}{=} \frac{\sec x+1}{\sec x-1}$$

$$\frac{\frac{1}{\cos x}+1}{\frac{1}{\cos x}-1}\left(\frac{\cos x}{\cos x}\right)$$

$$\frac{1+\cos x}{1-\cos x}$$

–EZ!

Example 8:
Prove the following identity:

$$\sin^2 x\cos^3 x \stackrel{?}{=} \left(\sin^2-\sin^4 x\right)\left(\cos x\right)$$

$$\sin^2 x\left(1-\sin^2 x\right)\cos x$$

$$\sin^2 x\cdot\cos^2 x\cdot\cos x$$

$$\sin^2 x\cos^3 x$$

–KaPow!

Example 9:
Prove the following identity:

$$\cos^5 x \stackrel{?}{=} \left(1-2\sin^2 x+\sin^4 x\right)\left(\cos x\right)$$

$(1-\sin^2 x)^2 \cdot \cos x$

$(\cos^2 x)^2 \cdot \cos x$

$\cos^4 x \cdot \cos x$

$\cos^5 x$

Math is Power

Example 10:
Prove the following identity:

$$\left(\sin x+\cos x\right)\left(\tan x+\cot x\right) \stackrel{?}{=} \sec x+\csc x$$

$\sin x \tan x + \sin x \cot x + \cos x \tan x + \cos x \cot x$

$\frac{\sin x}{1}\left(\frac{\sin x}{\cos x}\right) + \frac{\sin x}{1}\left(\frac{\cos x}{\sin x}\right) + \frac{\cos x}{1}\left(\frac{\sin x}{\cos x}\right) + \frac{\cos x}{1}\left(\frac{\cos x}{\sin x}\right)$

$\frac{\sin^2 x}{\cos x} + \cos x + \sin x + \frac{\cos^2 x}{\sin x}$

$\frac{\sin^3 x + \cos^2 x \sin x + \sin^2 x \cos x + \cos^3 x}{\sin x \cos x}$

A

$\frac{1}{\cos x} + \frac{1}{\sin x}$

$\frac{\sin x + \cos x}{\sin x \cos x}$

A

$\frac{\sin x(\sin^2 x + \cos^2 x) + \cos x(\sin^2 x + \cos^2 x)}{\sin x \cos x}$

$\frac{\sin x(1) + \cos x(1)}{\sin x \cos x}$

$\frac{\sin x + \cos x}{\sin x \cos x}$ – YES!

We'll conclude with a Kodiak Grizzly Bear!!

Example 11:
Prove the following identity:

$$\frac{\tan x}{1-\cot x}+\frac{\cot x}{1-\tan x} \stackrel{?}{=} 1+\sec x\csc x$$

* useful

$(x^3+y^3)=(x+y)(x^2-xy+y^2)$

$(x^3-y^3)=(x-y)(x^2+xy+y^2)$

$\frac{\frac{\sin x}{\cos x}}{1-\frac{\cos x}{\sin x}} + \frac{\frac{\cos x}{\sin x}}{1-\frac{\sin x}{\cos x}}$

$\frac{\frac{\sin x}{\cos x}}{1-\frac{\cos x}{\sin x}}\left(\frac{\sin x\cos x}{\sin x\cos x}\right) + \frac{\frac{\cos x}{\sin x}}{1-\frac{\sin x}{\cos x}}\left(\frac{\sin x\cos x}{\sin x\cos x}\right)$

$\frac{\sin^2 x}{\sin x\cos x - \cos^2 x} + \frac{\cos^2 x}{\sin x\cos x - \sin^2 x}$

$\frac{\sin^2 x}{\cos x(\sin x - \cos x)} - \frac{\cos^2 x}{\sin x(\sin x - \cos x)}$

$\frac{\sin^3 x - \cos^3 x}{\sin x \cdot \cos x \cdot (\sin x - \cos x)}$

$\frac{\cancel{(\sin x - \cos x)}(\sin^2 x + \sin x\cos x + \cos^2 x)}{\cancel{(\sin x - \cos x)}\sin x \cdot \cos x}$

$\frac{\sin x\cos x + 1}{\cos x \sin x}$

$1 + \frac{1}{\cos x}\cdot\frac{1}{\sin x}$

$\frac{\cos x \sin x + 1}{\cos x \sin x}$

– Grrr..