Chapter 6.3: Composite Identities

Have you ever wanted to do something, but you didn't think it was right?

Perhaps you wanted to say that $\ln(a+b)$ was equal to $\ln a + \ln b$. Of course it is not equal.

If you were asked to "expand" the expression $\cos(x+y)$, would you be tempted to say it was $\cos x + \cos y$? I hope your answer is "No!"

There is a way to do it, but not by "distributing" the cosine. The angle $x+y$ is called a **composite angle,** because it combines two angles x and y to form a new angle.

Composite Identities for Cosine

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

Notice the sign change. We can summarize both as

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

Example 1:
Find the simplified, exact value of $\cos 15^\circ$ by finding two angles from the Unit Circle that either add or subtract to give 15°. Verify on your calculator. $60^\circ - 45^\circ = 15^\circ$ or $225^\circ - 210^\circ$ or others

$\cos 15^\circ$
$\cos(60^\circ - 45^\circ)$
$\cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ$
$\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$
$\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$
$\boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$
≈ 0.9659

OR

$\cos 15^\circ$
$\cos(225^\circ - 210^\circ)$
$\cos 225^\circ \cos 210^\circ + \sin 225^\circ \sin 210^\circ$
$\left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)$
$\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$
$\boxed{\frac{\sqrt{2}+\sqrt{6}}{4}}$

Exact values of trig ratios of any angle that is a multiple of 15° or $\frac{\pi}{12}$ radians can be found using a composite of Unit Circle angles.

Example 2:

Find the simplified, exact value of $\cos\frac{7\pi}{12}$ by finding two angles from the Unit Circle that either add or subtract to give $\frac{7\pi}{12}$. Verify on your calculator. $\frac{7\pi}{12} = \frac{4\pi}{12} + \frac{3\pi}{12} = \frac{\pi}{3} + \frac{\pi}{4}$

$\cos\frac{7\pi}{12}$

$\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$

$\cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$

$\boxed{\frac{\sqrt{2}-\sqrt{6}}{4}}$

≈ -0.2588

Example 3:

Using the cofunction identity $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, derive an identity for $\sin(x+y)$.

$\sin(x+y)$

$\cos\left(\frac{\pi}{2}-(x+y)\right)$

$\cos\left(\frac{\pi}{2}-x-y\right)$

$\cos\left(\left(\frac{\pi}{2}-x\right)-y\right)$

$\cos\left(\frac{\pi}{2}-x\right)\cos y + \sin\left(\frac{\pi}{2}-x\right)\sin y$

$\sin x\cos y + \cos x\sin y$

or

$\sin x\cos y + \sin y\cos x$

Composite Identities for Sine

$$\sin(x+y) = \sin x\cos y + \sin y\cos x$$
$$\sin(x-y) = \sin x\cos y - \sin y\cos x$$

Notice the sign does NOT change. We can summarize both as

$$\sin(x \pm y) = \sin x\cos y \pm \sin y\cos x$$

Example 4:
Use the composite identities to prove the following cofunction identity:

$$\sin\left(\frac{\pi}{2}-x\right)=\cos x$$

$\sin\left(\frac{\pi}{2}-x\right) = \cos x$

$\sin\frac{\pi}{2}\cos x - \sin x\cos\frac{\pi}{2}$

$(1)\cos x - \sin x(0)$

$\cos x$ ✓

Example 5:

Find the simplified, exact value of $\sin\frac{35\pi}{12}$ by finding two angles from the Unit Circle that either add or subtract to give $\frac{35\pi}{12}$ (or an angle coterminal with it). Verify on your calculator.

$\sin\frac{35\pi}{12}$

$\sin\left(\frac{26\pi}{12}+\frac{9\pi}{12}\right)$

$\sin\left(\frac{13\pi}{6}+\frac{3\pi}{4}\right)$

$\sin\frac{13\pi}{6}\cos\frac{3\pi}{4}+\sin\frac{3\pi}{4}\cos\frac{13\pi}{6}$

$\sin\frac{\pi}{6}\cos\frac{3\pi}{4}+\sin\frac{3\pi}{4}\cos\frac{\pi}{6}$

Example 6:
Write each of the following as the sine or cosine of a single angle:

(a) $\sin 22^\circ\cos 13^\circ+\cos 22^\circ\sin 13^\circ$

$\sin(22^\circ+13^\circ)$

$\sin 35^\circ$

(b) $\sin x\sin 2x-\cos x\cos 2x$

$-\cos x\cos 2x+\sin x\sin 2x$

$-(\cos x\cos 2x-\sin x\sin 2x)$

$-\cos(x+2x)$

$-\cos 3x$

Composite Identities for Tangent

$$\tan(x \pm y) = \frac{\sin(x \pm y)}{\cos(x \pm y)} \quad \text{or} \quad \tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

Example 7:

Find the simplified, rationalized, exact value of $\tan\frac{5\pi}{12}$ using BOTH of the identities above. Verify on your calculator. $\frac{5\pi}{12} = \frac{8\pi}{12} - \frac{3\pi}{12}$

$\tan\frac{5\pi}{12}$

$\tan\left(\frac{8\pi}{12} - \frac{3\pi}{12}\right)$

$\tan\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$

$\dfrac{\tan\frac{2\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{2\pi}{3}\cdot\tan\frac{\pi}{4}}$

$\dfrac{-\sqrt{3}-1}{1+(-\sqrt{3})(1)}$

$\dfrac{-\sqrt{3}-1}{1-\sqrt{3}}$

$-\left(\dfrac{\sqrt{3}+1}{1-\sqrt{3}}\right)$

$\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\left(\dfrac{\sqrt{3}+1}{\sqrt{3}+1}\right)$

$\dfrac{3+2\sqrt{3}+1}{3-1}$

$\dfrac{4+2\sqrt{3}}{2}$

$\boxed{2+\sqrt{3}}$

≈ 3.732

$\dfrac{\sin\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)}{\cos\left(\frac{2\pi}{3}-\frac{\pi}{4}\right)}$

$\dfrac{\sin\frac{2\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\cos\frac{2\pi}{3}}{\cos\frac{2\pi}{3}\cos\frac{\pi}{4} + \sin\frac{2\pi}{3}\sin\frac{\pi}{4}}$

$\dfrac{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$

$\dfrac{(\sqrt{6}+\sqrt{2})/4}{(\sqrt{6}-\sqrt{2})/4}$

$\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}\left(\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}\right)$

$\dfrac{6+2\sqrt{12}+2}{6-2}$

$\dfrac{8+4\sqrt{3}}{4}$

$\boxed{2+\sqrt{3}}$

Example 8:
Prove the following identities:

(a) $\tan\left(\theta + \frac{\pi}{4}\right) \stackrel{?}{=} \frac{1+\tan\theta}{1-\tan\theta}$

$\dfrac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\cdot\tan\frac{\pi}{4}}$

$\dfrac{\tan\theta + 1}{1 - \tan\theta}$

$\dfrac{1+\tan\theta}{1-\tan\theta}$ — done

(b) $\sin(x-y) + \sin(x+y) \stackrel{?}{=} 2\sin x\cos y$

$\sin x\cos y - \cancel{\sin y\cos x} + \sin x\cos y + \cancel{\sin y\cos x}$

$\sin x\cos y + \sin x\cos y$

$2\sin x\cos y$ — done

Example 9:
Prove the following identity:

$$\sin 3u \stackrel{?}{=} 3\cos^2 u\sin u - \sin^3 u$$

$\sin(2u+u)$

$\sin 2u\cos u + \sin u\cos 2u$

$\sin(u+u)\cos u + \sin u\cos(u+u)$

$(\sin u\cos u + \sin u\cos u)\cos u + \sin u(\cos u\cos u - \sin u\sin u)$

$2\sin u\cos u\cdot\cos u + \sin u(\cos^2 u - \sin^2 u)$

$2\sin u\cos^2 u + \sin u\cos^2 u - \sin^3 u$

$3\sin u\cos^2 u - \sin^3 u$ — BAM!